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Monge–Ampère measures and geometric characterization of affine algebraic varieties

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Abstract. To every continuous plurisubharmonic exhaustion function φ on a Stein space, we associate a collection of positive measures supported on the level sets of φ, defined by means of the Monge–Ampère operators in the sense of Bedford and Taylor. We show that these measures play a fundamental role in the study of the growth and convexity properties of plurisubharmonic or holomorphic functions. When the Monge–Ampère volume of the variety is finite, a Siegel-type algebraicity theorem applies to holomorphic functions of φ-polynomial growth. We deduce that finiteness of the Monge–Ampère volume, together with a suitable lower bound on the Ricci curvature, is a necessary and sufficient geometric condition characterizing affine algebraic varieties.

Translator's note

This unofficial English translation of Jean-Pierre Demailly's 1985 memoir *Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines* was prepared using ChatGPT 5.6 Sol Ultra and subsequently revised by Yu-Chi Hou with the assistance of Claude Opus 5. It is based on both the TeX transcription available on the author's website and the published text in *Mémoires de la Société Mathématique de France*, no. 19 (1985). Any mathematically or textually significant change to either source is explained in a footnote at the relevant point.

The footnotes use a few short labels. [P] means that the wording or formula has been restored from the published 1985 text. [E] marks a correction to an error that appears in both the published text and the later web TeX. [W] indicates material found only in the web TeX that has been kept in this translation. [C] marks a passage where the published text and the web TeX differ, and explains which version is followed here. [P/E] is used when both restoration and editorial correction are involved. Minor changes in wording, spelling, punctuation, spacing, and typography are not recorded.

I would like to thank Chung-Ming Pan for pointing out several errors introduced during the AI-assisted translation. If you notice any further errors or typos, I would be grateful if you would let me know at `yhou1994@umd.edu`.

Contents

0 Introduction.

The present study is set in the general framework of complex analytic spaces. The first section is therefore devoted to defining differential forms, positive currents, and plurisubharmonic functions on a possibly singular complex space X. Given a local embedding of X into an open set $\Omega \subset \mathbb{C}^N$, we define differential forms on X as the restrictions to X of "ambient" forms on Ω; spaces of currents are then obtained by duality, as in the smooth case.

Definition 0.1. – *Let $V : X \to [-\infty, +\infty[$ be a function.*

(a) *V is said to be plurisubharmonic (psh for short) on X if V is locally the restriction to X of psh functions on the ambient space $\mathbb{C}^N$.*

(b) *V is said to be weakly psh if V is locally integrable and locally bounded above on X, and if $dd^cV \geqslant 0$.*

Notation. Throughout, we set

$$d^c = i(\overline{\partial} - \partial), \quad \textit{so that} \quad dd^c = 2i\,\partial\overline{\partial}.$$

Every psh function is then weakly psh, but in general a weakly psh function need not agree almost everywhere with a psh function. We nevertheless show that the two notions coincide when the space X is locally irreducible. The proof of this result uses two ingredients: on the one hand, the characterization of psh functions due to Fornaess and Narasimhan [FN], and on the other, an extension theorem for bounded psh functions across the singular locus of X (which uses resolution of singularities). We also study the behavior of closed positive currents and psh functions under proper direct image.

In §2, we essentially follow the method developed by Bedford and Taylor [BT2] to give meaning to the positive current $dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ when the φ_j are locally bounded psh functions, and we generalize it to the case in which one of the functions (say φ_i) is not locally bounded. The classical Chern-Levine-Nirenberg inequalities can then be stated as follows:

Theorem 0.2. – *For every open set $\omega \Subset X$ and every compact set $K \subset \omega$, there are constants C_1, C_2 depending only on ω and K such that the following mass estimates hold:*

(a) $\displaystyle\int_K \|dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k\| \leq C_1\|\varphi_1\|_{L^1(\omega)}\|\varphi_2\|_{L^\infty(\omega)} \cdots \|\varphi_k\|_{L^\infty(\omega)},$

(b) $\displaystyle\int_K \|\varphi_1\, dd^c\varphi_2 \wedge \ldots \wedge dd^c\varphi_k\| \leq C_2\|\varphi_1\|_{L^1(\omega)}\|\varphi_2\|_{L^\infty(\omega)} \cdots \|\varphi_k\|_{L^\infty(\omega)}.$

In this setting we finally establish the sequential continuity of the Monge-Ampère operators

$$(\varphi_1, \ldots, \varphi_k) \longmapsto dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k \quad \text{and} \quad \varphi_1\, dd^c\varphi_2 \wedge \ldots \wedge dd^c\varphi_k$$

for decreasing sequences $\varphi_1^\nu, \ldots, \varphi_k^\nu$ of psh functions.

Now suppose that X is a Stein space equipped with a continuous psh exhaustion function $\varphi : X \to [-\infty, R[$. We write

$$B(r) = \{z \in X\,;\ \varphi(z) < r\}, \qquad S(r) = \{z \in X\,;\ \varphi(z) = r\}, \qquad r \in [-\infty, R[$$

for the "pseudoballs" and "pseudospheres" associated with φ. We show that these data naturally determine a collection of positive measures μ_r, supported on the spheres $S(r)$, which we call the Monge-Ampère measures associated with φ. They are simply defined by

$$\mu_r(h) = \int_{S(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi, \qquad n = \dim X,$$

when φ is of class $\mathcal{C}^2$ and r is a regular value of φ. When φ is merely continuous, we must use the Bedford-Taylor definition of $(dd^c)^n$ and set

$$\mu_r = \big(dd^c \max(\varphi, r)\big)^n - \mathbf{1}_{X \smallsetminus B(r)} (dd^c\varphi)^n.$$

There is then a general Jensen-Lelong-type formula whose proof is an immediate consequence of Stokes' and Fubini's theorems (cf. §3).

Theorem 0.3. – *Every psh function V on X is μ_r-integrable for every $r < R$, and*

$$\int_{-\infty}^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} = \mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n.$$

We also show that the measures μ_r depend continuously on φ with respect to decreasing sequences. As in the $\mathcal{C}^\infty$ case, this shows that (μ_r) is the weakly left-continuous family of measures that disintegrates the positive current $(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi$ over the spheres $S(r)$.

The measures μ_r constructed in this way have a number of natural properties important in the study of the growth and convexity of psh functions.

Section 4 studies the "residual" measure $\mu_{-\infty} = \mathbf{1}_{S(-\infty)}(dd^c\varphi)^n$, supported on the polar set $S(-\infty)$. By (0.3), the measure $\mu_{-\infty}$ can also be defined as the weak limit of μ_r as r tends to $-\infty$. Following our earlier work [De4, De5], we show that

$\mu_{-\infty}$ essentially depends only on the asymptotic behavior of φ near $S(-\infty)$. This result yields the classical inequality[1]

$$(0.4)\qquad (dd^c\varphi)^n \geqslant (2\pi)^n \sum_{x\in X} \nu(\varphi,x)^n\,\delta_x,$$

where $\nu(\varphi,x)$ denotes the Lelong number of φ at a point $x\in X$, and δ_x is the Dirac measure at x (at a singular point x, this measure must be counted with multiplicity equal to the multiplicity of X at x).

In §5, we show that the measures μ_r satisfy the maximum principle with respect to psh functions, namely that for every psh function V one has

$$(0.5)\qquad \sup_{B(r)} V = \text{essential supremum of } V \text{ with respect to } \mu_r.$$

The remarkable point is that this equality holds even though the support of μ_r may be very sparse in $S(r)$, as happens, for example, when the pseudoballs $B(r)$ are analytic polyhedra.

Section 6 generalizes to the present setting the classical convexity properties due to P. Lelong concerning means of psh functions on balls, spheres, polydiscs, …. We show that the natural geometric hypothesis underlying these convexity properties is that φ be a solution of the homogeneous Monge-Ampère equation $(dd^c\varphi)^n\equiv 0$. More precisely:

Theorem 0.6. – *Suppose that $(dd^c\varphi)^n\equiv 0$ on the open set $\{\varphi>A\}$. Let V be a psh function on X. Then the supremum of V on $B(r)$, the means $\mu_r(V)$, and, more generally, the L^p-means $r\mapsto[\mu_r(V_+^p)]^{1/p}$ are increasing and convex functions of $r\in\,]A,R[$.*[2]

This result follows from elementary second-derivative calculations involving Jensen's formula 0.3 and Stokes' and Fubini's theorems. More generally, we prove a version of Theorem 0.6 "with parameters" for the measures $\mu_{y,r}$ on the fibers $\pi^{-1}(y)$ of a holomorphic fibration $\pi: X\to Y$. The psh function φ on X is assumed to be exhaustive on the fibers and to satisfy $(dd^c\varphi)^n\equiv 0$ on the open set $\{\varphi>A\}$, where n is the dimension of the fibers. Then the means $\mu_{y,r}(V)$ and the L^p-means are weakly psh functions of (y,z) on $Y\times\mathbb{C}$, where $r=\operatorname{Re} z$. This readily yields the following extension of Theorem 0.6 to product spaces.

Theorem 0.7. – *Let $X_1,\ldots,X_k$ be Stein spaces, equipped with continuous psh exhaustion functions $\varphi_j: X_j\to[-\infty,R_j[$ such that $(dd^c\varphi_j)^{n_j}\equiv 0$ on the open set*

[1] **[P]** The published version includes both $dd^c\varphi$ and the factor $(2\pi)^n$ in (0.4). The factor also agrees with Corollary 4.8.

[2] **[P]** The published version gives the interval $]A,R[$, as do Theorems 6.3 and 6.4.

$\{\varphi_j > A_j\}$, $n_j = \dim X_j$. If V is psh on $X_1 \times \cdots \times X_k$, then the L^p-mean

$$M_V^p(r_1, \ldots, r_k) = \Big[\mu_{r_1} \otimes \cdots \otimes \mu_{r_k}(V_+^p)\Big]^{1/p}$$

is jointly convex in the variables $(r_1, \ldots, r_k) \in \prod_{1 \leqslant j \leqslant k}]A_j, R_j[$.

In Sections 7 and 8, we make the additional assumption that the volume of X has moderate growth at infinity (here the "radius" R is assumed to equal $+\infty$). More precisely, we assume that

$$\lim_{r \to +\infty} \frac{1}{r} \|\mu_r\| = 0. \tag{0.8}$$

Under this hypothesis, Jensen's formula 0.3 implies the following fundamental inequality:

$$\int_X dd^c V \wedge (dd^c\varphi)^{n-1} \leqslant \liminf_{r \to +\infty} \frac{1}{r}\, \mu_r(V_+), \tag{0.9}$$

which yields a number of results concerning the growth of psh functions or the value distribution of holomorphic functions (as suggested by the paper of N. Sibony and P.-M. Wong [SW]). In particular, every bounded psh or holomorphic function on X is constant.

Given a holomorphic function f on X, we also define the "degree" of f relative to φ by

$$\delta_\varphi(f) = \limsup_{r\to+\infty} \frac{1}{r}\,\mu_r(\log_+|f|), \tag{0.10}$$

and say that f is φ-polynomial if $\delta_\varphi(f)$ is finite. Inequality (0.9) then implies that the order of vanishing of f at a regular point $a \in X$ satisfies the estimate

$$\mathrm{ord}_a(f) \leqslant C(a)\,\delta_\varphi(f).$$

An elementary linear-algebra argument due to Siegel consequently gives the following algebraicity theorem (we assume X irreducible).

Theorem 0.11. – *Let $K_\varphi(X)$ be the field of meromorphic functions of the form f/g, where f and g are φ-polynomial. Then:*

(a) $0 \leqslant \deg\mathrm{tr}_{\mathbb{C}}\, K_\varphi(X) \leqslant \dim_{\mathbb{C}} X$;

(b) *If* $\deg\mathrm{tr}_{\mathbb{C}}\, K_\varphi(X) = \dim_{\mathbb{C}} X$, *then the field $K_\varphi(X)$ is finitely generated.*

As a special case of this theorem, we recover W. Stoll's result [St1] characterizing algebraic varieties in $\mathbb{C}^N$ by the property that their area has minimal growth.

Part B of this work is devoted to characterizing affine algebraic varieties by an "intrinsic" geometric criterion involving finiteness of the Monge-Ampère volume and a lower bound on the Ricci curvature. More precisely, we prove the following result:

Theorem 0.12. – *Let X be a smooth, connected complex analytic variety of dimension n. Then X is analytically isomorphic to an affine algebraic variety X_{alg} if and only if X satisfies condition* (c) *below and possesses a strictly psh exhaustion function φ of class $\mathcal{C}^\infty$ such that:*

(a) $\mathrm{Vol}(X) = \int_X (dd^c\varphi)^n < +\infty$;

(b) *The Ricci curvature of the metric $\beta = dd^c(e^\varphi)$ has a lower bound of the form*

$$\mathrm{Ricci}(\beta) \geqslant -\frac{1}{2}dd^c\psi,$$

with $\psi \in \mathcal{C}^\infty(X,\mathbb{R})$, where $\psi \leqslant A\varphi + B$ and A, B are constants $\geqslant 0$;

(c) *The even-degree cohomology spaces $H^{2q}(X,\mathbb{R})$ are finite-dimensional.*

The ring of regular functions of the algebraic structure X_{alg} is then given by the intersection $K_\varphi(X) \cap \mathcal{O}(X)$.

Following W. Stoll's work on strictly parabolic varieties (cf. [St2] and [Bu]), D. Burns posed the problem of characterizing affine algebraic varieties in terms of exhaustion functions having special properties, for example satisfying the homogeneity condition $(dd^c\varphi)^n \equiv 0$ outside a compact set. Theorem 0.12 gives a partial answer to this problem. It also belongs to the line of sufficient conditions obtained by Mok, Siu, and Yau [SY], [MSY], [Mok1,2,3], although our hypotheses differ substantially from those in the works just cited. The broad outline of our argument is in fact analogous to the approach followed in [Mok1,2,3].

Section 10 proves the necessity of conditions 0.12 (a,b,c) for every algebraic set $X \subset \mathbb{C}^N$. The function φ is then given by $\varphi(z) = \log(1 + |z|^2)$, so that the metric $dd^c\varphi$ coincides with the Fubini-Study metric on projective space $\mathbb{P}^N$. By an explicit computation of the Ricci curvature, we verify that the curvature inequality (b) holds with $\psi = \log \sum_{K,L} |J_{K,L}|^2$, where the $J_{K,L}$ are the Jacobian determinants associated with a system of polynomial equations for X. We also show by a counterexample that curvature condition (b) is indeed indispensable.

The proof that conditions (a,b,c) are sufficient occupies §11,12,13. Its general outline is as follows. Using the Hörmander-Nakano-Skoda L^2 estimates for the $\overline{\partial}$ operator and hypothesis (b), we construct a system $F = (f_1, \ldots, f_N)$ of φ-polynomial holomorphic functions which, outside an analytic set $S \subset X$, defines an embedding of $X \setminus S$ into $\mathbb{C}^N$. Under the finite-volume hypothesis (a), algebraicity theorem 0.11 implies that the transcendence degree of $f_1, \ldots, f_N$ is $n = \dim X$. Thus the morphism F maps X into an algebraic variety $M \subset \mathbb{C}^N$ of dimension n.

The principal remaining difficulty is to prove that the embedding is quasi-surjective, that is, that the open set $\Omega = F(X \setminus S)$ is a Zariski-open subset of M. We obtain this result by first showing that the $(1,1)$-form $F_*(dd^c\varphi)$ extends to a closed positive current T of finite mass on M, with $T = 0$ on $M \setminus \Omega$. In view of the mass estimates arising from the construction, this is done essentially by the integration-by-parts method developed by H. Skoda [Sk5] and H. El Mir [EM]. To indicate the rest of the argument, consider the already significant case $N = n$, i.e. $M = \mathbb{C}^n$. There is then a psh function V on $\mathbb{C}^n$ of minimal growth, i.e. $V(z) \leqslant C_1 \log_+ |z| + C_0$, such that $dd^cV = T$. By construction, the function $\tau = V - \varphi \circ F^{-1}$ is pluriharmonic on Ω and τ tends to $-\infty$ at every point of $\partial\Omega$. It follows that the closed set $M \setminus \Omega$ is pluripolar. To show that $M \setminus \Omega$ is in fact algebraic, our method is to verify, using algebraicity theorem 0.11 once again, that the holomorphic 1-form $h = \partial\tau$ extends to a rational meromorphic form on $M = \mathbb{C}^n$.

When M is an arbitrary affine algebraic variety, the link that exists in $\mathbb{C}^n$ between closed positive $(1,1)$-currents T of finite projective mass and psh functions of minimal growth no longer applies. Nevertheless, one can show (see the appendix, §15) that the differential inequality $dd^cV \geqslant T$ can always be solved on M by a psh solution V such that the current $dd^cV - T$ is of class $\mathcal{C}^\infty$ and has polynomial growth. The end of the proof is then almost identical. Hypothesis (c), for its part, is used to show that the "Zariski topology" on X is quasi-compact, and hence that X can be covered by finitely many open sets of the form $X \smallsetminus S$ (cf. §13). We do not in fact know whether hypothesis (c) is truly indispensable.

Finally, Theorem 0.12 extends to complex spaces with isolated singularities (cf. §9), but the extension to the general case raises difficulties that will be studied in §14.

A. Monge-Ampère measures and growth of plurisubharmonic functions

1 Currents and plurisubharmonic functions on complex spaces.

The purpose of this section is to define currents and psh functions on a possibly singular complex analytic space. Readers wishing to consider only the smooth case in what follows may proceed directly to §2.

Let X be a reduced complex space of pure dimension n, and let X_{reg} (resp. X_{sing}) be the set of its regular (resp. singular) points. Since the definitions we shall consider are local, we may, without restriction, assume that X is identified with a closed analytic subset of an open set $\Omega \subset \mathbb{C}^N$ by means of an embedding $j : X \to \Omega$.

We then define the space $\mathcal{C}^k_{p,q}(X)$ of (p,q)-forms of class $\mathcal{C}^k$ on X, $k \in \mathbb{N} \cup \{\infty\}$, as the image of the restriction morphism

$$j^* : \mathcal{C}^k_{p,q}(\Omega) \to \mathcal{C}^k_{p,q}(X_{\mathrm{reg}}),$$

equipped with the quotient topology. If $j_1 : X \to \Omega_1 \subset \mathbb{C}^{N_1}$ is another embedding, there are (locally) holomorphic maps $f : \Omega \to \mathbb{C}^{N_1}$ and $g : \Omega_1 \to \mathbb{C}^N$ such that $j_1 = f \circ j$ and $j = g \circ j_1$. The commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "j_1"'] & f^{-1}(\Omega_1) \subset \Omega \arrow[d, "\mathrm{Id} \times f"] \\ \Omega_1 \supset g^{-1}(\Omega) \arrow[r, "g \times \mathrm{Id}"'] & \Omega \times \Omega_1, \end{tikzcd}$$

then shows that the morphisms j and j_1 do indeed induce the same image space $\mathcal{C}^k_{p,q}(X)$, since $\mathrm{Id} \times f$ and $g \times \mathrm{Id}$ are closed smooth embeddings.

Definition 1.1. – *We denote by $\mathcal{D}_{p,q}(X)$ (resp. $\mathcal{D}^k_{p,q}(X)$) the space of compactly supported (p,q)-forms on X of class $\mathcal{C}^\infty$ (resp. $\mathcal{C}^k$), equipped with the inductive limit topology. By definition, the dual space $\mathcal{D}'_{p,q}(X)$ is the space of currents of bidimension (p,q) and bidegree $(n-p, n-q)$ on X. Currents belonging to the subspace $[\mathcal{D}^k_{p,q}(X)]'$ are said to be currents of order k.*

If $T \in [\mathcal{D}^k_{p,q}(X)]'$, the current $j_*T \in [\mathcal{D}^k_{p,q}(\Omega)]'$ defined by

$$\langle j_*T, v\rangle = \langle T, j^*v\rangle$$

for every form $v \in \mathcal{D}^k_{p,q}(\Omega)$ is supported on $j(X)$. Nevertheless, when $k \geqslant 1$, a current $\theta \in [\mathcal{D}^k_{p,q}(\Omega)]'$ supported on $j(X)$[3] need not come from a current T defined on X, even if X is smooth.

The usual differential operators d, ∂, $\overline{\partial}$, and the operation of exterior multiplication by a $\mathcal{C}^\infty$ form are also extended to currents by duality, exactly as in the smooth case. It would be particularly interesting to be able to compute in general the local cohomology groups of the operators d and $\overline{\partial}$; in fact, we do not even know whether these groups always vanish for arbitrary singularities.

Definition 1.2. – *A current $T \in \mathcal{D}'_{p,p}(X)$ is said to be (weakly) positive if the current of bidegree (n,n)*

$$T \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$$

is a measure $\geqslant 0$ for every system of $(1,0)$-forms $(\alpha_1, \ldots, \alpha_p)$ of class $\mathcal{C}^\infty$ on X.

Equivalently, the current j_*T is $\geqslant 0$ on Ω; in particular, a current $T \geqslant 0$ on X is necessarily of order 0.

Now let $F : X \to Y$ be a morphism of analytic spaces X, Y of dimensions n, m, respectively. To ensure that the pullback morphism

$$F^* : \mathcal{C}^k_{p,q}(Y) \to \mathcal{C}^k_{p,q}(X)$$

is well defined, it suffices to verify the following lemma:

Lemma 1.3. – *Let $j : Y \to \Omega \subset \mathbb{C}^N$ be an embedding and $\alpha \in \mathcal{C}^k_{p,q}(\Omega)$ a form such that $\alpha_{|Y_{\mathrm{reg}}} = 0$. Then $F^*\alpha_{|X_{\mathrm{reg}}} = 0$.*[4]

Proof. We may assume that X is smooth and connected. If $F(X) \not\subset Y_{\mathrm{sing}}$, then $F^{-1}(Y_{\mathrm{reg}})$ is dense in X, and the result follows by continuity. Thus the only difficulty is the case $F(X) \subset Y_{\mathrm{sing}}$. Arguing by induction on the dimension of Y and factoring F as

$$X \xrightarrow{F} Y_{\mathrm{sing}} \hookrightarrow Y,$$

we see that it is enough to replace F by the inclusion morphism $Y_{\mathrm{sing}} \hookrightarrow Y$. Lemma 1.3 then follows from the continuity of α and the following lemma:

[3] **[E]** The support must be $j(X)$. The expression $j(\Omega)$ in the web TeX cannot be correct because j is defined on X, not on Ω.

[4] **[P]** Here the regular locus is X_{reg}. The lowercase x_{reg} in the web TeX is a typo.

Lemma 1.4. – *Let a be a regular point of $Y_{\rm sing}$. Then there is a sequence of points $\{a_\nu\} \subset Y_{\rm reg}$ converging to a such that, in the Grassmannian of m-planes in $\mathbb{C}^N$, the tangent space $T_{a_\nu} Y_{\rm reg}$ converges to a plane containing $T_a Y_{\rm sing}$.*

Proof. This is a consequence of the existence of Whitney stratifications of Y; see [Wh1] and [Wh2]. □

Suppose that the morphism $F : X \to Y$ is proper. We then define the direct-image map

$$F_* : [\mathcal{D}^k_{p,q}(X)]' \longrightarrow [\mathcal{D}^k_{p,q}(Y)]'$$

by duality, setting, for every current $T \in [\mathcal{D}^k_{p,q}(X)]'$ and every form $\alpha \in \mathcal{D}^k_{p,q}(Y)$,

$$\langle F_* T, \alpha \rangle = \langle T, F^* \alpha \rangle.$$

If T is $\geqslant 0$, then clearly so is $F_* T$. Moreover, the direct-image morphism F_* commutes with the operators d, d^c, ∂, $\overline{\partial}$. Thus, if T is closed and $\geqslant 0$, then $F_* T$ is also closed and $\geqslant 0$.

We now turn to the definition of psh functions.

Definition 1.5. – *Let $V : X \to [-\infty, +\infty[$ be a function that is not identically $-\infty$ on any open subset of X. We say that V is plurisubharmonic on X (psh for short) if, for every local embedding $j : X \hookrightarrow \Omega \subset \mathbb{C}^N$, V is locally the restriction of a psh function on Ω.*

J.E. Fornaess and R. Narasimhan gave the following fundamental characterization of psh functions on a complex space.

Theorem 1.6 ([FN], th. 5.3.1) – *A function $V : X \to [-\infty, +\infty[$ is psh on X if and only if:*

(a) *V is upper semicontinuous*;

(b) *For every holomorphic map $f : \Delta \to X$ from the unit disc into X, $V \circ f$ is subharmonic or identically equal to $-\infty$ on Δ.*

This result readily gives the following generalization of Brelot's extension theorem to complex spaces.

Theorem 1.7. – *Let X be a locally irreducible complex space and $Y \subset X$ an analytic subset with empty interior in X. Let V be a psh function on $X \smallsetminus Y$, locally bounded above near Y. Then there is a unique psh function V^*[5] on X*

[5] **[P]** The published version denotes the extension by V^*, in agreement with the rest of Theorem 1.7.

extending V, given by

$$V^*(y) = \limsup_{x\in X\smallsetminus Y,\, x\to y} V(x), \qquad y \in Y.$$

Proof.

(a) *Uniqueness of* V^*. Since V^* is upper semicontinuous, for every $y \in Y$ we have

$$V^*(y) = \limsup_{x\in X\smallsetminus Y,\, x\to y} V^*(x) \geqslant \limsup_{x\in X\smallsetminus Y,\, x\to y} V(x).$$

Conversely, choose a holomorphic map $f : \Delta \to X$ such that $f(0) = y$ and $f(\Delta) \not\subset Y$. Then 0 is isolated in $f^{-1}(Y)$, and since $V^* \circ f$ is psh on Δ, we obtain

$$V^*(y) = V^*(f(0)) = \limsup_{t\neq 0,\, t\to 0} V(f(t)) \leqslant \limsup_{x\in X\smallsetminus Y,\, x\to y} V(x).$$

(b) *Plurisubharmonicity of* V^*. The result is local on X. By Hironaka's desingularization theorem [Hi], there are a smooth space X' and a proper modification $\sigma : X' \to X$; by definition, σ is proper and induces, outside an analytic set $Z \subset X$, an isomorphism

$$\sigma : X' \smallsetminus \sigma^{-1}(Z) \xrightarrow{\sim} X \smallsetminus Z.$$

For every $x \in X$, the fiber $\sigma^{-1}(x)$ is compact and connected. Indeed, if $\sigma^{-1}(x)$ were disconnected, the point x would have an irreducible open neighborhood U (X is assumed locally irreducible) such that $\sigma^{-1}(U)$ was disconnected; but then $U \smallsetminus Z$ would be connected and $\sigma^{-1}(U) \smallsetminus \sigma^{-1}(Z)$ disconnected, a contradiction. The function $V\circ\sigma$ is psh on $X' \smallsetminus \sigma^{-1}(Y)$ and locally bounded above near $\sigma^{-1}(Y)$. By Brelot's theorem in the smooth case, $V \circ \sigma$ extends to a psh function V' on X'. The function V' is necessarily constant on the fibers $\sigma^{-1}(x)$, and therefore descends to an upper semicontinuous function V^* on X. We now show that V^* is psh on X by using Theorem 1.6.[6] Given a germ of a holomorphic map $f : (\Delta, 0) \to (X, x)$, there is in X' a germ of a curve (Γ', x') lying over the image $\Gamma = f(\Delta)$; hence there are an integer $k \in \mathbb{N}^*$ and a germ $f' : (\Delta, 0) \to (X', x')$ such that $f(t^k) = \sigma(f'(t))$. Consequently $V^*(f(t^k)) = V'(f'(t))$ is psh on $(\Delta, 0)$, which implies that $V^* \circ f$ is also psh. □

Proposition 1.8. – *Every psh function V on X is locally integrable with respect to the area measure of X (relative to any embedding $j : X \to \Omega \subset \mathbb{C}^N$).*

Proof. Since V is locally bounded above by definition, we may assume $V \leqslant 0$. After rotating the coordinates if necessary, at every point of X there is a local embedding $j : X \hookrightarrow P$ into a polydisc $P \subset \mathbb{C}^N$ such that the projections $\pi^I : X \to P^I$ onto

[6] **[E]** The correct reference is Theorem 1.6, whose characterization is used here. Both source versions incorrectly refer back to Theorem 1.7.

the coordinate n-planes with coordinates z_j, $j \in I$, $I \subset \{1, \ldots, N\}$, $|I| = n$, are proper with finite fibers. Thus, for every I, there is an analytic set $S_I \subset P^I$ such that the restriction $\pi^I : X \smallsetminus (\pi^I)^{-1}(S_I) \to P^I \smallsetminus S_I$ is a finite covering. The function $\pi^I_* V$ defined by[7]

$$\pi^I_* V(y) = \sum_{x \in (\pi^I)^{-1}(y)} V(x)$$

is psh $\leqslant 0$ on $P^I \smallsetminus S_I$, and hence extends to a psh function V_I on all of P^I. Since the area measure of X is given by

$$d\sigma_X = \sum_{|I|=n} (\pi^I)^* d\lambda_{\mathbb{C}^n},$$

where $d\lambda$ is Lebesgue measure, the conclusion follows from the local integrability of the V_I on P^I. $\square$

Proposition 1.8 shows that every psh function on X can be regarded as a current of bidegree $(0, 0)$. By regularizing a local extension of V to $\mathbb{C}^N$ and passing to the decreasing limit, one readily checks that the $(1,1)$-current $dd^c V = 2i\partial\overline{\partial} V$ is positive on X.

Definition 1.9. – *A locally integrable function V on X is said to be weakly psh if V is locally bounded above and $dd^c V \geqslant 0$ in the sense of currents.*

Unlike the smooth case, the assumption that V be locally bounded above is essential. For example, consider the curve parametrized by $(z_1, z_2) = (t^2, t^3)$ in $\mathbb{C}^2$; the function $V(t) = \mathrm{Re}(1/t)$[8] is not locally bounded above at 0, yet one may verify that $dd^c V = 0$ in the sense of currents (cf. Definition 1.1). Observe also that a weakly psh function need not agree almost everywhere with a psh function, as shown by the function on the curve $z_1 z_2 = 0$ in $\mathbb{C}^2$ defined by $V(z_1, 0) = 1$, $V(0, z_2) = 0$ if $z_2 \neq 0$. We nevertheless have the following result:

Theorem 1.10. – *Given a function $V : X \to [-\infty, +\infty[$, the following properties* (a), (b), (c) *are equivalent:*

(a) *V is weakly psh on X.*

(b) *V agrees almost everywhere with a psh function V_{reg} on X_{reg}, locally bounded above near X_{sing}.*

[7] **[P]** The direct-image function is $\pi^I_* V$. The corresponding expression in the web TeX is garbled.

[8] **[P]** The published version has $\mathrm{Re}(1/t)$. The variable t was dropped from the web TeX.

(c) *There exists a psh function* $\tilde{V}$[9] *on the normalization* $\tilde{X}$ *of* X *such that* $\tilde{V} = V \circ \pi$ *almost everywhere, where* $\pi : \tilde{X} \to X$ *is the natural morphism.*

For V *to be equal almost everywhere to a psh function on* X*, it is necessary and sufficient that the following condition hold: for every* $a \in X$*, let* $V^*(a)$ *be the essential upper limit of* $V(x)$ *as* $x \in X$ *tends to* a*, and let* (X_j, a) *be the irreducible components of the germ* (X, a)*; then*

$$\limsup_{x \in X_j\,;\, x \to a} \text{ess}\ V(x) = V^*(a), \quad \forall j.$$

Under this hypothesis V^* *is psh on* X *and* $V = V^*$ *almost everywhere.*

Proof. (a) $\Rightarrow$ (b). This implication follows immediately from Definition 1.9 and the well-known smooth case.

(b) $\Rightarrow$ (a). Let $h_1 = \ldots = h_m = 0$ be local equations of X_{sing} in X. Then for every $\varepsilon > 0$ the function

$$V_\varepsilon = \begin{cases} V_{\text{reg}} + \varepsilon \log\left(|h_1|^2 + \cdots + |h_m|^2\right) & \text{on } X_{\text{reg}} \\ -\infty & \text{on } X_{\text{sing}} \end{cases}$$

is psh on X by Theorem 1.6. Thus $dd^c V_\varepsilon \geqslant 0$. Since $dd^c V_\varepsilon$ converges weakly to $dd^c V$ as ε tends to 0, it follows that $dd^c V \geqslant 0$.

(b) $\Rightarrow$ (c). The function $V_{\text{reg}} \circ \pi$ is psh on $\tilde{X} \smallsetminus \pi^{-1}(X_{\text{sing}}) = \pi^{-1}(X_{\text{reg}})$, locally bounded above near $\pi^{-1}(X_{\text{sing}})$, and $\tilde{X}$ is locally irreducible. Theorem 1.7 shows that $V_{\text{reg}} \circ \pi$ extends to a psh function $\tilde{V}$ on $\tilde{X}$.

(c) $\Rightarrow$ (b) follows from the fact that $\pi : \tilde{X} \smallsetminus \pi^{-1}(X_{\text{sing}}) \to X_{\text{reg}}$ is an isomorphism.

As for the last assertion, the condition we have given for the plurisubharmonicity of V^* is clearly necessary. To see that it is sufficient, observe that the set of irreducible components (X_j, a) is in bijective correspondence with the set of points a_j of $\tilde{X}$ lying over a (this follows, for example, from Narasimhan [Nar], Prop. VI.2), and that

$$\tilde{V}(a_j) = \limsup_{x \in X_j\,,\, x \to a} \text{ess}\ V(x).$$

Thus, by hypothesis, $V^* \circ \pi = \tilde{V}$ at every point of X; since every holomorphic map $f : \Delta \to X$ lifts to a map $\tilde{f} : \Delta \to \tilde{X}$, the plurisubharmonicity of $\tilde{V}$ implies that of V^*. □

Corollary 1.11. – *If* X *is locally irreducible and* V *is weakly psh on* X*, then the function defined by*

$$V^*(a) = \limsup_{x \to a} \text{ess}\ V(x), \qquad a \in X,$$

[9] **[P]** The function on the normalization is $\widetilde{V}$. The web TeX omits the symbol V.

is psh on X and $V = V^*$ almost everywhere.

Corollary 1.12. – *If $V : X \to [-\infty, +\infty[$ is continuous and weakly psh, then V is psh.*

To conclude this section, we examine the behavior of psh functions under direct image.

Proposition 1.13. – *Let $F : X \to Y$ be a proper surjective morphism with finite fibers.*

(a) *If V is a weakly psh function on X, the function F_*V defined by*

$$F_*V(y) = \sum_{x \in F^{-1}(y)} V(x)$$

is weakly psh on Y, and moreover

$$dd^c(F_*V) = F_*(dd^cV).$$

(b) *Suppose in addition that Y is locally irreducible.*[10] *If V is psh and the sum defining F_*V, $\sum_{x \in F^{-1}(y)} V(x)$, is counted with multiplicities, then F_*V is psh on Y.*

Proof.

(a) We know that F is a branched covering, i.e. there is an analytic set $Z \subset Y$ such that

$$F : X \smallsetminus F^{-1}(Z) \to Y \smallsetminus Z$$

is a covering of smooth varieties. It follows that the definition of F_*V agrees with the definition given for direct images of currents. It is clear that F_*V is locally bounded above on Y, and the property $dd^c(F_*V) = F_*dd^cV \geqslant 0$ follows from the fact that F_* commutes with the operators d and d^c.

(b) Under the hypothesis that Y is locally irreducible, the cardinality of the fiber $F^{-1}(y)$, $y \in Y \smallsetminus Z$, is locally constant near a point of Z. If, in addition, V is continuous, F_*V extends continuously across Z to Y, and Corollary 1.12 shows that F_*V is psh. In the general case, for every fiber $F^{-1}(y) = \{x_1, \ldots, x_m\}$ there are arbitrarily small neighborhoods O_j of x_j, $1 \leqslant j \leqslant m$, and a neighborhood U of

[10] **[E]** Local irreducibility is required for the target Y, not for X; this is exactly what the proof uses to show that F_*V is psh on Y.

y such that $F^{-1}(U) = O_1 \cup \ldots \cup O_m$. Write V as a decreasing limit of continuous psh functions V_k on such a neighborhood $F^{-1}(U)$.[11] Then

$$F_* V = \lim_{k \to +\infty} F_* V_k \quad \text{on } U,$$

and hence $F_* V$ is psh. □

2 The operators $(dd^c)^k$ and the Chern-Levine-Nirenberg inequalities.

In this section, we recall the definition of the Monge-Ampère operators $(dd^c)^k$ introduced by Bedford and Taylor [BT1], [BT2]. This definition gives meaning to the current $dd^c V_1 \wedge \ldots \wedge dd^c V_k$ when the V_j are bounded psh functions. Here we shall need to consider the slightly more general case in which one of the functions V_j need not be bounded, and in this setting we give a proof of the Chern-Levine-Nirenberg inequalities [CLN]. Finally, as in [BT2], we study the continuity of the operator $(dd^c)^k$ with respect to decreasing limits of psh functions.

Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be locally bounded psh functions on X, and let V be an arbitrary psh function. By Bedford-Taylor [BT2], the closed current $dd^c V \wedge dd^c \varphi_1 \wedge \ldots \wedge dd^c \varphi_k$, which is $\geqslant 0$, can be defined recursively in k by setting

$$dd^c V \wedge dd^c \varphi_1 \wedge \ldots \wedge dd^c \varphi_k = dd^c\big(\varphi_k \, dd^c V \wedge dd^c \varphi_1 \wedge \ldots \wedge dd^c \varphi_{k-1}\big). \tag{2.1}$$

The positivity of the right-hand side is clear by the induction hypothesis if $\varphi_k \in \mathcal{C}^\infty(X)$; the general case follows by regularizing φ_k and passing to the weak limit in the space of currents.

Let $\Omega = \{\rho < 0\}$ be a relatively compact open subset of X, defined by a strictly psh $\mathcal{C}^\infty$ function ρ on a neighborhood Ω' of Ω, with $d\rho \neq 0$ on $\partial\Omega$. For every real number $a > 0$ and every integer $0 \leqslant k \leqslant n$, set

$$\beta_k = |\rho|^{k+a} (dd^c \rho)^{n-k} + (k+a)|\rho|^{k-1+a} d\rho \wedge d^c \rho \wedge (dd^c \rho)^{n-k-1}$$

and denote by $\|v\|_p$ the L^p norm of a function v on Ω with respect to the measure β_0, $p \in [1, +\infty]$. The mass of the current (2.1) then satisfies the following estimates (cf. [CLN]).

Theorem 2.2. *Let V, $V_1, \ldots, V_k$ be psh functions on X such that $V \leqslant 0$ and $V_j \geqslant 0$, $1 \leqslant j \leqslant k$, on Ω. Then there are constants $C_j = C_j(k, a) \geqslant 0$, $j = 1, 2, 3$, such that*

[11] **[P]** The neighborhood here is $F^{-1}(U)$. The expression $F^{-1}(V)$ in the web TeX refers to no defined set V.

(a) $\displaystyle\int_\Omega \beta_{k+1} \wedge dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k \leqslant C_1 \,\|V\|_1 \,\|\varphi_1\|_\infty \cdots \|\varphi_k\|_\infty.$

(b) $\displaystyle\int_\Omega \beta_k \wedge |V|\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k \leqslant C_2 \,\|V\|_1 \,\|\varphi_1\|_\infty \cdots \|\varphi_k\|_\infty.$

(c) $\displaystyle\int_\Omega \beta_k \wedge dd^c V_1 \wedge \ldots \wedge dd^c V_k \leqslant C_3 \,\|V_1\|_k \,\|V_2\|_k \cdots \|V_k\|_k.$

Proof. By Lemma 2.4 below, we may assume that the V_j and φ_j are of class $\mathcal{C}^\infty$. A direct computation gives

$$\begin{aligned} d^c\beta_k &= -2(k+a)\,|\rho|^{k-1+a}\, d^c\rho \wedge (dd^c\rho)^{n-k},\\ dd^c\beta_k &= 2(k+a)\Big[-|\rho|^{k-1+a}\,(dd^c\rho)^{n-k+1} + (k-1+a)\, d\rho\wedge d^c\rho \wedge (dd^c\rho)^{n-k}\Big], \end{aligned}$$

and hence the inequality of forms

$$|dd^c\beta_k| \leqslant 2(k+a)\beta_{k-1}.$$

Let I_k, J_k denote the integrals in (a), (b), respectively, and set $\psi_k = dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$. By the integration-by-parts formula of Lemma 2.5, and since

$$\beta_{k+1|\partial\Omega} = d^c\beta_{k+1|\partial\Omega} = 0,$$

we obtain

$$\begin{aligned} I_k &= \int_\Omega dd^c\beta_{k+1} \wedge \varphi_k\, dd^c V \wedge \psi_{k-1}\\ &\leqslant 2(k+1+a)\,\|\varphi_k\|_\infty \int_\Omega \beta_k \wedge dd^c V \wedge \psi_{k-1} = 2(k+1+a)\,\|\varphi_k\|_\infty I_{k-1},\\ I_0 &= \int_\Omega V\, dd^c\beta_1\\ &\leqslant 2(1+a)\int_\Omega |V|\,|\rho|^a (dd^c\rho)^n \leqslant 2(1+a)\,\|V\|_1. \end{aligned}$$

This proves (a) by induction with $C_1(k,a) = 2^{k+1}(1+a)\ldots(k+1+a)$; the inequality in fact remains valid even when $a=0$. On the other hand, if $k \geqslant 1$, we obtain

$$\begin{aligned} J_k &= \int_\Omega -\varphi_k dd^c(V\beta_k) \wedge \psi_{k-1}\\ &= \int_\Omega -\varphi_k\big(V\, dd^c\beta_k + \beta_k \wedge dd^c V + 2\, dV \wedge d^c\beta_k\big) \wedge \psi_{k-1}\\ &= \int_\Omega \varphi_k\big(V\, dd^c\beta_k - \beta_k \wedge dd^c V\big) \wedge \psi_{k-1} + 2V\, d\varphi_k \wedge d^c\beta_k \wedge \psi_{k-1} \end{aligned}$$

after integrating by parts the term $-2\varphi_k\, dV \wedge d^c\beta_k \wedge \psi_{k-1}$ in the second line. We now suppose $\varphi_k > 0$ and apply the Cauchy-Schwarz inequality to the component of bidegree $(n-k+1, n-k+1)$ of the current

$$2d\varphi_k \wedge d^c\beta_k = -4(k+a)\,|\rho|^{k-1+a} d\varphi_k \wedge d^c\rho \wedge (dd^c\rho)^{n-k},$$

which gives the upper bound

$$\Big[4(k+a)^2\varphi_k|\rho|^{k-2+a}d\rho\wedge d^c\rho+|\rho|^{k+a}\frac{d\varphi_k\wedge d^c\varphi_k}{\varphi_k}\Big]\wedge(dd^c\rho)^{n-k}.$$

It follows that

$$\begin{aligned}J_k\leqslant\int_\Omega\varphi_k|V|\Big[-dd^c\beta_k+4(k+a)^2|\rho|^{k-2+a}d\rho\wedge d^c\rho\wedge(dd^c\rho)^{n-k}\Big]\wedge\psi_{k-1}\\+\int_\Omega|V|\,|\rho|^{k+a}(dd^c\rho)^{n-k}\wedge\frac{d\varphi_k\wedge d^c\varphi_k}{\varphi_k}\wedge\psi_{k-1}.\end{aligned}$$

The form in brackets in the first integral equals

$$2(k+a)\Big[|\rho|^{k-1+a}(dd^c\rho)^{n-k+1}+(k+1+a)|\rho|^{k-2+a}d\rho\wedge d^c\rho\wedge(dd^c\rho)^{n-k}\Big]\leqslant C_4\beta_{k-1}$$

with $C_4=C_4(k,a)=\frac{2(k+a)(k+1+a)}{k-1+a}$. We therefore finally obtain

$$J_k\leqslant C_4\|\varphi_k\|_\infty J_{k-1}+\int_\Omega|V|\,\beta_k\wedge\psi_{k-1}\wedge\frac{d\varphi_k\wedge d^c\varphi_k}{\varphi_k}.$$

Let J_k' denote the integral obtained by replacing φ_k by $\varphi_k'=\exp(B\varphi_k)$ in J_k, and set $M=\|\varphi_k\|_\infty$. Then[12]

$$dd^c\varphi_k'=e^{B\varphi_k}\big(B\,dd^c\varphi_k+B^2\,d\varphi_k\wedge d^c\varphi_k\big)\geqslant B\,e^{-BM}dd^c\varphi_k+\frac{d\varphi_k'\wedge d^c\varphi_k'}{\varphi_k'},$$

$$B\,e^{-BM}J_k\leqslant J_k'-\int_\Omega|V|\,\beta_k\wedge\psi_{k-1}\wedge\frac{d\varphi_k'\wedge d^c\varphi_k'}{\varphi_k'}\leqslant C_4(k,a)\,e^{BM}J_{k-1}.$$

Since $\inf_{B>0}\frac{1}{B}e^{2BM}=2eM=2e\,\|\varphi_k\|_\infty$, this completes the proof of (b) by induction on k, with constant

$$C_2(k,a)=(4e)^k\,\frac{k+a}{a}\,(2+a)\cdots(k+1+a).$$

To prove (c), first suppose that $V_1=V_2=\ldots=V_k=v\geqslant 0$. Since

$$\Big(dd^cv^{\frac{k}{k-1}}\Big)^{k-1}\geqslant v\Big(\frac{k}{k-1}\,dd^cv\Big)^{k-1},$$

integration by parts gives

$$\int_\Omega\beta_k\wedge(dd^cv)^k=\int_\Omega dd^c\beta_k\wedge v\,(dd^cv)^{k-1}\leq 2(k+a)\Big(\frac{k-1}{k}\Big)^{k-1}\int_\Omega\beta_{k-1}\wedge\Big(dd^cv^{\frac{k}{k-1}}\Big)^{k-1}.$$

[12] **[P]** The published formula has $dd^c\varphi_k'$. The subscript k was lost in the web TeX.

By induction on k, this inequality in turn implies

$$\int_\Omega \beta_k \wedge (dd^c v)^k \leqslant 2^k(1+a)...(k+a)\frac{(k-1)!}{k^{k-1}}\int_\Omega \beta_0 v^k.$$

Now replace v by

$$v = \frac{V_1}{\|V_1\|_k} + \cdots + \frac{V_k}{\|V_k\|_k}.$$

Then

$$\frac{k!}{\|V_1\|_k \cdots \|V_k\|_k}\int_\Omega \beta_k \wedge dd^c V_1 \wedge \ldots \wedge dd^c V_k \leqslant 2^k(1+a)...(k+a)\frac{(k-1)!}{k^{k-1}}\int_\Omega \beta_0 v^k$$

while $\|v\|_k \leqslant k$. Inequality (c) follows with

$$C_3(k,a) = 2^k(1+a)\ldots(k+a). \qquad \square$$

An immediate consequence of Theorem 2.2 (b) is that the current $V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ has locally finite mass on X; in particular, we recover the following result due to Bedford-Taylor [BT2].

Corollary 2.3. – *The coefficient measures of $dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ do not charge the pluripolar sets $\{V = -\infty\}$.* $\square$

Since the space X is Stein, by J.E. Fornaess and R. Narasimhan [FN] there are decreasing sequences V_m, $\varphi_{j,m}$ of $\mathcal{C}^\infty$ psh functions on X such that

$$V_m \to V, \qquad \varphi_{j,m} \to \varphi_j \qquad \text{for } 1 \leqslant j \leqslant k.$$

Lemma 2.4. – *There are strictly increasing sequences of integers $m(\nu)$, $m_1(\nu)$, $\ldots$, $m_k(\nu)$, $\nu \in \mathbb{N}$, such that, in the sense of weak convergence of measures, either one of the following convergence properties may be obtained:*

(a) $dd^c V_{m(\nu)} \wedge dd^c\varphi_{1,m_1(\nu)} \wedge \ldots \wedge dd^c\varphi_{k,m_k(\nu)} \longrightarrow dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$

(b) $V_{m(\nu)}\, dd^c\varphi_{1,m_1(\nu)} \wedge \ldots \wedge dd^c\varphi_{k,m_k(\nu)} \longrightarrow V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$.[13]

Proof. By Theorem 2.2, already proved for $\mathcal{C}^\infty$ psh functions, the sequences (a), (b) are locally bounded in mass, and bounded subsets of the space of currents of order 0 are metrizable for the weak topology. In case (a), observe that

$$\varphi_{k,m} dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \longrightarrow \varphi_k dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$$

[13] **[P]** Here $V_{m(\nu)}$ and V multiply the currents that follow them; there should be no wedge sign immediately after either scalar factor.

by monotone convergence of $\varphi_{k,m}$ as $m \to +\infty$; applying dd^c, we therefore have

$$dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\varphi_{k,m} \longrightarrow dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\varphi_k.$$

Since the topology is metrizable, we may successively choose $m_k(\nu) = \nu$ and then $m_{k-1}(\nu)$, …, $m_1(\nu)$, $m(\nu)$ by induction on ν (resp. $m(\nu) = \nu$ and then $m_1(\nu)$, …, $m_k(\nu)$ in case (b)) to obtain the desired convergence.[14] □

For later reference, we state here the integration-by-parts lemma used above.

Lemma 2.5. – *If u and v are forms of class $\mathcal{C}^2$ of respective bidegrees (p,q) and $(n-p-1, n-q-1)$, with $p+q$ even, then*[15]

$$\int_\Omega u \wedge dd^cv = \int_\Omega dd^cu \wedge v + \int_{\partial\Omega} \big(u \wedge d^cv - d^cu \wedge v\big).$$

Indeed, it suffices to apply Stokes' theorem to the form

$$d\big(u \wedge d^cv - d^cu \wedge v\big) = u \wedge dd^cv - dd^cu \wedge v + du \wedge d^cv + d^cu \wedge dv$$

and observe that

$$du \wedge d^cv = i(\partial u \wedge \overline{\partial} v - \overline{\partial} u \wedge \partial v) = -d^cu \wedge dv. \qquad \square$$

Adapting the techniques of [BT2] to the present setting, we now prove the continuity of the operator $(dd^c)^k$ with respect to decreasing limits of psh functions.

Theorem 2.6. – *Let $\{\varphi_j^\nu\}_{\nu\in\mathbb{N}} \subset L^\infty_{\mathrm{loc}}(X)$ and $\{V^\nu\}_{\nu\in\mathbb{N}}$ be decreasing sequences of psh functions such that*

$$\varphi_j = \lim_{\nu\to+\infty} \varphi_j^\nu \in L^\infty_{\mathrm{loc}}(X), \qquad V = \lim_{\nu\to+\infty} V^\nu \not\equiv -\infty.$$

Then, in the sense of weak convergence of measures,

(a) $dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \longrightarrow dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$,

(b) $V^\nu\, dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \longrightarrow V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$,

(c) $\varphi_k^\nu dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \longrightarrow \varphi_k\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$.[16]

[14] **[P]** The last index in the diagonal sequence is $m_k(\nu)$, as in the published proof. The indices are shifted in the web TeX.

[15] **[P]** The published version contains the full three-term integration-by-parts identity. The web TeX omits the interior term $\int_\Omega dd^cu \wedge v$.

[16] **[P]** The scalar factors V^ν and φ_k^ν are not followed by wedge signs. In part (c), the last indexed function is φ_{k-1}.

We shall prove Theorem 2.6 simultaneously with the following property, which is a corollary of it.

Corollary 2.7. –

(a) $dd^c\big(V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k\big) = dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$.[17]

(b) *The currents* $V\, dd^c\varphi_1\wedge\ldots\wedge dd^c\varphi_k$ *and* $dd^cV\wedge dd^c\varphi_1\wedge\ldots\wedge dd^c\varphi_k$ *are symmetric in* $\varphi_1, \ldots, \varphi_k$.

Proof. Using Lemma 2.4 and the standard diagonal-sequence argument, we reduce to the case in which V^ν, φ_1^ν, …, φ_k^ν are of class $\mathcal{C}^\infty$. Since properties 2.6 (a,b,c) are local, we may, without loss of generality, work in an open set $\Omega = \{\rho < 0\} \Subset X$. We now reduce to the situation in which φ_j, φ_j^ν are of class $\mathcal{C}^\infty$ near $\partial\Omega$ and vanish on $\partial\Omega$, so that Stokes' formula can be applied without boundary terms. Let $\mathbb{R}^2 \ni (u,v) \mapsto \lambda(u,v)$ be a $\mathcal{C}^\infty$ function that is convex and increasing in u and v, and agrees with $\max(u,v)$ for $|u-v| > 1$. Then $\tilde\varphi_j^\nu = \lambda(\varphi_j^\nu - \frac{2}{\varepsilon}, \varepsilon^{-2}\rho)$ is psh and $\mathcal{C}^\infty$; moreover, for $\varepsilon > 0$ sufficiently small,

$$\begin{cases} \tilde\varphi_j^\nu = \varphi_j^\nu - \frac{2}{\varepsilon} & \text{on } \ \Omega_{3\varepsilon} = \{\rho < -3\varepsilon\}, \\ \tilde\varphi_j^\nu = \varepsilon^{-2}\rho & \text{on } \ \overline{\Omega} \smallsetminus \Omega_\varepsilon = \{-\varepsilon \leqslant \rho \leqslant 0\}. \end{cases}$$

We may therefore ultimately assume that $\varphi_j^\nu = \varphi_j = \varepsilon^{-2}\rho$ on the "annulus" $\overline{\Omega}\smallsetminus\Omega_\varepsilon$ (for all j, ν).

Proof of 2.6 (a). We argue by induction on k. By (2.1), it is enough to prove that[18]

$$\begin{aligned} &\lim_{\nu\to+\infty} \int_\Omega \varphi_k^\nu\, dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \wedge dd^c\psi \\ &\qquad = \int_\Omega \varphi_k\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\psi \end{aligned} \tag{2.8}$$

for every form $\psi \in \mathcal{C}^\infty_{n-k-1,n-k-1}(\overline{\Omega})$ such that $\psi_{|\partial\Omega} = 0$ (note that by hypothesis $\varphi^\nu_{k|\partial\Omega} = 0$). By successively replacing ψ by $\rho(dd^c\rho)^{n-k-1}$ and $\psi + A\rho(dd^c\rho)^{n-k-1}$, $A \gg 0$, we may assume $dd^c\psi \geqslant 0$ on $\overline{\Omega}$. The inequality $\leqslant$ in (2.8) then follows directly from the induction hypothesis

$$dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \longrightarrow dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$$

[17] **[P]** Inside this current, V is a scalar factor and is therefore not joined to the following form by a wedge sign.

[18] **[P]** Formula (2.8) follows the published version, which supplies the missing factors and indices and gives the boundary term the correct sign.

and the monotone convergence theorem. To prove the reverse inequality $\geqslant$, we perform successive integrations by parts using Lemma 2.5:

$$\begin{aligned}
&\int_\Omega \varphi_k\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\psi \\
&\qquad\leqslant \int_\Omega \varphi_k^\nu\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\psi \\
&\qquad= \int_\Omega \varphi_{k-1}\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-2} \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\
&\qquad\leqslant \int_\Omega \varphi_{k-1}^\nu\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-2} \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\
&\qquad= \cdots \leqslant \int_\Omega \varphi_1^\nu\, dd^cV \wedge dd^c\varphi_2^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\
&\qquad= \int_\Omega V\, dd^c\varphi_1^\nu \wedge dd^c\varphi_2^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\
&\qquad\qquad - \int_{\partial\Omega} V\, d^c\varphi_1^\nu \wedge dd^c\varphi_2^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\
&\qquad\leqslant \int_\Omega V^\nu\, dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi - \varepsilon^{-2k} \int_{\partial\Omega} V\, d^c\rho \wedge (dd^c\rho)^{k-1} \wedge dd^c\psi \\
&\qquad= \int_\Omega \varphi_k^\nu\, dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \wedge dd^c\psi \\
&\qquad\qquad + \varepsilon^{-2k} \int_{\partial\Omega} (V^\nu - V)\, d^c\rho \wedge (dd^c\rho)^{k-1} \wedge dd^c\psi.
\end{aligned}$$

The last integral tends to 0 by monotone convergence, which completes the proof of 2.6 (a).

Proof of 2.7 (a). Immediate consequence of 2.4 (b) and 2.6 (a).

Proof of 2.6 (b). Inequality 2.2 (b) implies that the sequence $V^\nu\, dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu$ has locally uniformly bounded mass on X. Moreover, every cluster value T of this sequence satisfies

$$T \leqslant V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k,$$

with equality on $\Omega \smallsetminus \Omega_\varepsilon$ (where $\varphi_j^\nu = \varphi_j = \varepsilon^{-2}\rho$). On the other hand, by 2.6 (a) and 2.7 (a),

$$dd^cT = dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k = dd^c\big(V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k\big).$$

We now distinguish two cases according to the value of the integer k.

If $k \leqslant n-1$, Lemma 2.5 applied with $v = \rho(dd^c\rho)^{n-k-1}$ implies that the positive current

$$u = V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k - T$$

vanishes on Ω.

If $k = n$, we may regard V, φ_1, …, φ_k as functions on $X \times \mathbb{C}$ independent of the last variable, and apply the already established result 2.6 (b) to $X \times \mathbb{C}$. The proof of 2.6 (c) is identical. □

Remark 2.9. – If φ_j is $\geqslant 0$, we may write

$$d\varphi_j \wedge d^c\varphi_j = \frac{1}{2} dd^c\varphi_j^2 - \varphi_j\, dd^c\varphi_j;$$

consequently, the weak convergence theorem 2.6 remains valid for any product of V or dd^cV by $(1,1)$-forms of the type $dd^c\varphi_j$, $d\varphi_j \wedge d^c\varphi_j$, or, by polarization, $d\varphi_i \wedge d^c\varphi_j + d\varphi_j \wedge d^c\varphi_i$.

Remark 2.10. – The reader will find an interesting discussion of the definition and continuity of the Monge-Ampère operator in [Ki] and [Ce]. In particular, some of the preceding results can be extended to the case in which the functions φ_j are no longer necessarily bounded, provided one assumes that the poles of the φ_j are contained in a compact set. We shall assume that there is a compact set $K \subset X$ such that $\varphi_1, \ldots, \varphi_k$ are locally bounded on $X \smallsetminus K$. Then Definition (2.1), Lemma 2.4 (a), and Theorem 2.6 (a) remain valid.

To see this, observe that the problem arises only near K. Let ρ be a strictly psh $\mathcal{C}^\infty$ function and ω an open set such that $K \subset \omega \Subset \Omega = \{\rho < 0\} \Subset X$. After replacing φ_j by

$$\begin{cases} \varphi_j - \frac{2}{\varepsilon} & \text{on } \omega, \\ \varepsilon^{-2}\rho & \text{on } X \smallsetminus \Omega, \\ \max\left(\varphi_j - \frac{2}{\varepsilon}, \varepsilon^{-2}\rho\right) & \text{on } \Omega \smallsetminus \omega, \end{cases}$$

we may assume that $\varphi_j = \varepsilon^{-2}\rho$ near $\partial\Omega$. We first prove by induction that $\varphi_k\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$ (and hence also $dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$) has locally finite mass if $k \leqslant n-1$. Indeed, for every $a < 0$, with the notation $\varphi_{k,a} = \max(\varphi_k, a)$, we have:

$$\begin{aligned}\int_\Omega &|\varphi_{k,a}|\, dd^cV\wedge dd^c\varphi_1\wedge\ldots\wedge dd^c\varphi_{k-1}\wedge(dd^c\rho)^{n-k}\\ &\qquad= \int_\Omega |\rho|\, dd^c\big(\varphi_{k,a}dd^cV\wedge dd^c\varphi_1\wedge\ldots\wedge dd^c\varphi_{k-1}\big)\wedge(dd^c\rho)^{n-k-1}\\ &\qquad\le C\int_\Omega dd^c\big(\varphi_{k,a}dd^cV\wedge dd^c\varphi_1\wedge\ldots\wedge dd^c\varphi_{k-1}\big)\wedge(dd^c\rho)^{n-k-1}\\ &\qquad= C\varepsilon^{-2k}\int_\Omega dd^cV\wedge(dd^c\rho)^{n-1}<+\infty,\end{aligned}$$

where the last equality follows from Stokes' theorem. The proofs of 2.4 (a) and 2.6 (a) then go through without any change.

3 Monge-Ampère measures and Jensen's formula.

To every continuous psh exhaustion function φ on a Stein space, we shall canonically associate a family of positive measures supported on the level sets of φ. These measures arise naturally when one seeks to extend Jensen's formula to several variables. The principal ideas of this section are based on the calculations made by P. Lelong [Le1] to prove the existence of the Lelong numbers of a closed positive current. We use essentially the notation of [De4], [De5] (see also the paper of H. Skoda [Sk5]).[19]

Let X be a reduced Stein space of pure dimension n, equipped with a continuous psh function $\varphi : X\to[-\infty,R[$, where $R\in\,]-\infty,+\infty]$. For every $r<R$, denote by

$$B(r)=\{z\in X\,;\ \varphi(z)<r\},\qquad \overline{B}(r)=\{z\in X\,;\ \varphi(z)\leqslant r\}$$

the open and closed "pseudoballs" associated with φ (note that $\overline{B}(r)$ is not necessarily the closure of $B(r)$!).[20] We assume that φ is *exhaustive*, that is, that the pseudoballs $\overline{B}(r)$, $r<R$, are *compact*. Finally, for every $r\in[-\infty,R[$, set

$$\begin{aligned}&S(r)=\{z\in X\,;\ \varphi(z)=r\}=\overline{B}(r)\smallsetminus B(r),\\ &\varphi_r=\max(\varphi,r),\qquad \alpha=dd^c\varphi=2i\,\partial\overline{\partial}\varphi.\end{aligned}$$

The current $(dd^c\varphi_r)^n$ is well defined by (2.1) if $r>-\infty$, and if $r=-\infty$, $(dd^c\varphi_r)^n=(dd^c\varphi)^n$ exists by Remark 2.10.

[19] **[P]** The published paper cites [Sk5] here. The web TeX instead cites [Sk1], which is a different paper.

[20] **[P]** The closed pseudoball is $\overline{B}(r)=\{\varphi\leqslant r\}$, as in the published text. This notation is used consistently below.

Lemma 3.1. – *The map $r \mapsto (dd^c\varphi_r)^n$ is continuous from $[-\infty, R[$ into the space of measures on X equipped with the weak topology.*

Proof. Right continuity follows from Theorem 2.6 (a), while left continuity is obtained by writing

$$(dd^c\varphi_r)^n = \big(dd^c \max(\varphi - r, 0)\big)^n. \qquad \square$$

Since $(dd^c\varphi_r)^n$ vanishes on $B(r)$ and agrees with $(dd^c\varphi)^n$ on $X \smallsetminus \overline{B}(r)$, left continuity implies

$$(dd^c\varphi_r)^n \geqslant \mathbf{1}_{X \smallsetminus B(r)}(dd^c\varphi)^n,$$

where $\mathbf{1}_A$ denotes the characteristic function of a subset $A \subset X$. The results below follow immediately from these remarks and justify the following definition.

Theorem and Definition 3.2. – *The families of positive measures (μ_r), $(\overline{\mu}_r)$ supported on $S(r)$, $r \in [-\infty, R[$, defined by*

$$\mu_r = (dd^c\varphi_r)^n - \mathbf{1}_{X \smallsetminus B(r)}(dd^c\varphi)^n,$$
$$\overline{\mu}_r = (dd^c\varphi_r)^n - \mathbf{1}_{X \smallsetminus \overline{B}(r)}(dd^c\varphi)^n.$$

are called the Monge-Ampère measures associated with φ. The family μ_r (resp. $\overline{\mu}_r$) is weakly left-continuous (resp. weakly right-continuous), and the following relations hold:

$$\overline{\mu}_r = \lim_{\rho \to r+0} \mu_\rho, \qquad \mu_r = \lim_{\rho \to r-0} \overline{\mu}_\rho,$$
$$\overline{\mu}_r = \mathbf{1}_{S(r)}(dd^c\varphi_r)^n = \mu_r + \mathbf{1}_{S(r)}(dd^c\varphi)^n.$$

Let $D_\varphi \subset [-\infty, R[$ be the at most countable set of real numbers r such that $S(r)$ has positive measure[21] *with respect to $(dd^c\varphi)^n$ on X. Then $\overline{\mu}_r = \mu_r$ for every $r \notin D_\varphi$, and the maps $r \mapsto \mu_r$, $r \mapsto \overline{\mu}_r$ are continuous at every point $r \notin D_\varphi$.* $\square$

We are deeply grateful to E. Bedford for suggesting this definition, which simplifies the one used in an earlier version of this work. At every point where φ is regular, μ_r and $\overline{\mu}_r$ can be described by a simple differential form on the hypersurface $S(r)$.

Proposition 3.3. – *Let $x \in X$ be a regular point near which φ is of class $\mathcal{C}^2$ and such that $d\varphi(x) \neq 0$. Orient $S(r)$ as the boundary of $B(r)$. Then the measures μ_r and $\overline{\mu}_r$ are defined near x by the volume $(2n-1)$-form $(dd^c\varphi)^{n-1} \wedge d^c\varphi_{|S(r)}$.*[22]

[21] **[E]** The level set must have positive $(dd^c\varphi)^n$-measure. If its measure were zero, the jump displayed immediately above could not occur.

[22] **[P]** The published formula includes the factor $d^c\varphi$, which is necessary for the displayed $(2n-1)$-form.

Proof. Let Ω be a neighborhood of x on which $d\varphi \neq 0$, and let h be a $\mathcal{C}^\infty$ function with compact support in Ω. Write

$$\max(r,t) = \lim_{\nu\to+\infty} \chi_\nu(t)$$

where χ_ν is a sequence of convolution regularizations of $t \mapsto \max(r,t)$. We may arrange that (χ_ν) is a decreasing sequence of convex $\mathcal{C}^\infty$ functions such that $0 \leqslant \chi'_\nu \leqslant 1$, with $\lim \chi'_\nu(t)$ equal to 0 for $t < r$ and to 1 for $t > r$. Theorem 2.6 (a) therefore gives

$$\begin{aligned}
\int_\Omega h\,(dd^c\varphi_r)^n &= \lim_{\nu\to+\infty} \int_\Omega h\,(dd^c\chi_\nu\circ\varphi)^n \\
&= \lim_{\nu\to+\infty} -\int_\Omega dh\wedge(dd^c\chi_\nu\circ\varphi)^{n-1}\wedge d^c(\chi_\nu\circ\varphi) \\
&= \lim_{\nu\to+\infty} -\int_\Omega \chi'_\nu(\varphi)^n\,dh\wedge(dd^c\varphi)^{n-1}\wedge d^c\varphi \\
&= -\int_{\Omega\smallsetminus B(r)} dh\wedge(dd^c\varphi)^{n-1}\wedge d^c\varphi \\
&= \int_{\Omega\cap S(r)} h\,(dd^c\varphi)^{n-1}\wedge d^c\varphi + \int_{\Omega\smallsetminus B(r)} h\,(dd^c\varphi)^n
\end{aligned}$$

by Stokes' formula. Thus, on Ω, we obtain the equality of measures

$$(dd^c\varphi_r)^n = (dd^c\varphi)^{n-1}\wedge d^c\varphi_{|S(r)} + \mathbf{1}_{\Omega\smallsetminus B(r)}(dd^c\varphi)^n. \qquad \square$$

We can now prove the Jensen-Lelong formula announced above.

Theorem 3.4. – *Let V be a psh function on X. Then V is μ_r-integrable for every $r \in\,]-\infty, R[$. Moreover,*

$$\int_{-\infty}^r dt \int_{B(t)} dd^cV\wedge\alpha^{n-1} = \mu_r(V) - \int_{B(r)} V\,\alpha^n,$$

where $\alpha = dd^c\varphi$. Both sides are finite if $\inf_X \varphi > -\infty$ or if $\inf_{B(r)} V > -\infty$.

Proof. The integrability of V with respect to μ_r (and with respect to $\overline{\mu}_r$) follows from the integrability of V with respect to $(dd^c\varphi_r)^n$ by Theorem 2.2 (b). Also note that the integrals $\int_{B(t)} dd^cV\wedge(dd^c\varphi)^{n-1} \geqslant 0$ and $\int_{B(r)} V\,(dd^c\varphi)^n$ are well defined by Remark 2.10, the first always being convergent. The second converges if $\inf_X \varphi > -\infty$ by 2.2 (b), or if $\inf_{B(r)} V > -\infty$ by 2.10. To prove formula 3.4, first suppose

$$\inf_X \varphi > -\infty \quad \text{and} \quad \inf_{B(r)} V > -\infty,$$

and fix $c > r$. Fubini's theorem implies[23]

$$\begin{aligned}(3.5)\qquad \int_{-\infty}^{c} dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} &= \int_{B(c)} \Big[\int_{\{\varphi<t<c\}} dt\Big]\, dd^cV \wedge (dd^c\varphi)^{n-1}\\ &= \int_{B(c)} (c-\varphi)\, dd^cV \wedge (dd^c\varphi)^{n-1}.\end{aligned}$$

By Stokes' formula,

$$\begin{aligned}&\int_{B(c)} d\Big[(c-\varphi)\, d^cV \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^{n-1} \wedge d^c\varphi\Big]\\ &\qquad = \int_{B(c)} d\Big[(c-\varphi_r)\, d^cV \wedge (dd^c\varphi_r)^{n-1} + V\,(dd^c\varphi_r)^{n-1} \wedge d^c\varphi\Big]\end{aligned}$$

since the currents being integrated agree on the annulus $B(c) \smallsetminus \overline{B}(r)$. Expanding the first integral gives[24]

$$\int_{B(c)} (c-\varphi)\, dd^cV \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^n + \int_{B(c)} (dV \wedge d^c\varphi - d\varphi \wedge d^cV) \wedge (dd^c\varphi)^{n-1}$$

and, since the $(1,1)$-component of $dV \wedge d^c\varphi - d\varphi \wedge d^cV$ vanishes, the last integral vanishes. Consequently,

$$\int_{B(c)} (c-\varphi)\, dd^cV \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^n = \int_{B(c)} (c-\varphi_r)\, dd^cV \wedge (dd^c\varphi_r)^{n-1} + V\,(dd^c\varphi_r)^n.$$

Now let c tend to r from the right. Since $0 \leqslant c - \varphi_r \leqslant c - r$, the limit gives

$$\int_{\overline{B}(r)} (r-\varphi)\, dd^cV \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^n = \int_{\overline{B}(r)} V\,(dd^c\varphi_r)^n = \overline{\mu}_r(V).$$

In view of (3.5) and the identity $\overline{\mu}_r = \mu_r + \mathbf{1}_{S(r)}(dd^c\varphi)^n$, this proves formula 3.4 under the restrictive hypothesis that φ and V are bounded below. In the general case, write $V = \lim_{\nu\to+\infty} V_\nu$, where V_ν is a decreasing sequence of $\mathcal{C}^\infty$ psh functions on X (cf. [FN]). Fix $a < r$. By the preceding argument, after replacing φ by the locally bounded function φ_a, we obtain

$$\mu_r(V_\nu) = \int_a^r dt \int_{B(t)} dd^cV_\nu \wedge (dd^c\varphi_a)^{n-1} + \int_{B(r)} V_\nu\,(dd^c\varphi_a)^n.$$

Passing to the limit as ν tends to $+\infty$ gives

$$\mu_r(V) = \int_a^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi_a)^{n-1} + \int_{B(r)} V\,(dd^c\varphi_a)^n;$$

[23] **[P]** This identity is numbered (3.5) in the published paper. The number is restored because the proof refers to it later.

[24] **[P]** The expanded identity in the published proof includes the missing factor dd^cV.

indeed, the measure $dd^cV_\nu \wedge (dd^c\varphi_a)^{n-1}$ converges weakly to $dd^cV \wedge (dd^c\varphi_a)^{n-1}$ by Theorem 2.6 (a), and this measure is evaluated on the continuous function $\mathbf{1}_{B(r)}(r-\varphi_a)$ by (3.5). Stokes' theorem shows that the measure

$$dd^cV \wedge \big[(dd^c\varphi_a)^{n-1} - (dd^c\varphi)^{n-1}\big]$$

has integral zero over $B(t)$ for every $t > a$, and therefore

$$\int_a^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} = \mu_r(V) - \int_{B(r)} V\,(dd^c\varphi_a)^n.$$

This formula implies that the continuous functions $a \mapsto \int_{B(r)} V_\nu\,(dd^c\varphi_a)^n$ are increasing on $[-\infty, r[$. Their decreasing limit $a \mapsto \int_{B(r)} V\,(dd^c\varphi_a)^n$ is therefore right continuous, which allows us to pass to the limit at $a = -\infty$. □

Formula 3.4 immediately gives the analogous formula for the measures $\overline{\mu}_r$:

$$\int_{-\infty}^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} = \overline{\mu}_r(V) - \int_{\overline{B}(r)} V\,(dd^c\varphi)^n. \tag{3.6}$$

In particular, taking $V = 1$ gives:

Corollary 3.7. – *The total masses of μ_r and $\overline{\mu}_r$ are given by*

$$\|\mu_r\| = \int_{B(r)} (dd^c\varphi)^n = \int_{B(r)} \alpha^n, \qquad \|\overline{\mu}_r\| = \int_{\overline{B}(r)} (dd^c\varphi)^n = \int_{\overline{B}(r)} \alpha^n.$$

Throughout the remainder of the paper, we leave it to the reader to translate the results obtained into the case of the measures $\overline{\mu}_r$. We now study the dependence of the measures μ_r on the exhaustion φ.

Proposition 3.8. – *Let $(\varphi^\nu)_{\nu\in\mathbb{N}}$ be a decreasing sequence of continuous psh functions converging to φ on X, and let μ_r^ν be the Monge-Ampère measures associated with φ^ν. Then μ_r^ν converges weakly to μ_r for every $r \in\,]-\infty, R[\, \smallsetminus D_\varphi$.*

Proof. It suffices to apply Definition 3.2, which gives

$$\mu_r^\nu = (dd^c\varphi_r^\nu)^n - \mathbf{1}_{X\smallsetminus B^\nu(r)}(dd^c\varphi^\nu)^n$$

with

$$\varphi_r^\nu = \max(\varphi^\nu, r), \qquad B^\nu(r) = \{z \in X\,;\ \varphi^\nu(z) < r\},$$

and to observe that $B(r) = \bigcup B^\nu(r)$. Theorem 2.6 (a) then implies

$$(dd^c\varphi^\nu)^n \to (dd^c\varphi)^n, \qquad (dd^c\varphi_r^\nu)^n \to (dd^c\varphi_r)^n.$$ □

The following proposition shows that the measures μ_r are essentially the disintegration measures of the current $(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi$ over the family of pseudospheres $S(r)$.

Proposition 3.9. – *Let h be a bounded Borel function with compact support in the open set $X \smallsetminus S(-\infty)$. Then*

(a) $\displaystyle\int_{-\infty}^{R} \mu_r(h)\,dr = \int_X h\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi.$

(b) *If, in addition, h is of class $\mathcal{C}^1$, then*

$$\mu_r(h) = \int_{B(r)} \big(h\,\alpha^n + dh \wedge d^c\varphi \wedge \alpha^{n-1}\big).$$

Proof. (a) Both sides define positive measures acting on h. It is therefore enough to prove the equality when h is continuous with compact support. By Proposition 3.3 and Fubini's theorem, the formula holds when φ is of class $\mathcal{C}^\infty$: indeed, Sard's theorem shows that the set of critical values of φ is negligible. The general case follows by applying Proposition 3.8 to a sequence φ^ν of regularizations of φ.

(b) By 3.3, the formula holds if φ is $\mathcal{C}^\infty$ and r is not a critical value of φ. Proposition 3.8 extends the result to the case where φ is merely continuous, provided that $r \notin D_\varphi$. It remains only to observe that both sides are left-continuous functions of r.

□

Let $\chi :]-\infty, R[\to \mathbb{R}$ be a nonconstant increasing convex function. The measures μ_r^* associated with the exhaustion $\varphi^* = \chi \circ \varphi$ are related to the measures μ_r by the following change-of-variables formula:

Proposition 3.10. – *For every $r \in]-\infty, R[$, one has*

$$\mu^*_{\chi(r)} = \chi'_-(r)^n\,\mu_r, \qquad \overline{\mu}^*_{\chi(r)} = \chi'_+(r)^n\,\overline{\mu}_r$$

where χ'_+, χ'_- are the right and left derivatives of χ, respectively.

Proof. The equalities follow from Proposition 3.3 when φ, χ are of class $\mathcal{C}^\infty$ and r is a regular value of φ. Proposition 3.8 gives the general case for $r \notin D_\varphi$, after passage to a decreasing limit in φ and χ. The result follows by continuity for $r \in D_\varphi$. □

4 Residual measure of $(dd^c\varphi)^n$ on $S(-\infty)$.

If V is a psh function $\geqslant 0$, Theorem 3.4 shows that the function $r \mapsto \mu_r(V)$ is increasing and $\geqslant 0$. Moreover, since the double integral

$$\int_{-\infty}^{r} dt \int_{B(t)} dd^c V \wedge \alpha^{n-1} \leq \mu_r(V)$$

converges, we obtain

$$\lim_{r\to-\infty} \mu_r(V) = \lim_{r\to-\infty} \int_{B(r)} V\,\alpha^n = \int_{S(-\infty)} V\,\alpha^n.$$

Theorem and Definition 4.1. – *The measure $\overline{\mu}_{-\infty} = \mathbf{1}_{S(-\infty)}\alpha^n$ supported on the compact set $S(-\infty)$ is called the residual measure associated with φ. For every psh function $V \geqslant 0$ on X,*

$$\overline{\mu}_{-\infty}(V) = \lim_{r\to-\infty} \mu_r(V),$$

and μ_r converges weakly to $\overline{\mu}_{-\infty}$ as $r \to -\infty$.[25]

The last assertion follows from Theorem 3.2, or from the fact that every function h of class $\mathcal{C}^2$ on the Stein space X can be written as $h = h_1 - h_2$, with h_1, $h_2 \geqslant 0$ psh and of class $\mathcal{C}^2$.

The purpose of this section is to state some general properties of the residual measures $\overline{\mu}_{-\infty}$. To evaluate $\overline{\mu}_{-\infty}$ in concrete examples, we have the following comparison theorem, inspired by the results of [De4], [De5] on Lelong numbers.

Theorem 4.2. – *Let $\varphi_j : X \to [-\infty, R_j[$, $j = 1, 2$, be two continuous psh exhaustion functions, and let $\mu_{r,j}$ be the respective associated measures on $S_j(r) = \{\varphi_j = r\}$. Set*

$$\ell = \liminf_{\varphi_1(z)\to-\infty} \frac{\varphi_2(z)}{\varphi_1(z)}.$$

Then, for every psh function $V \geqslant 0$,

$$\overline{\mu}_{-\infty,2}(V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(V).$$

In particular, if $\varphi_2 \sim \varphi_1$ as $\varphi_1(z) \to -\infty$, then

$$\overline{\mu}_{-\infty,2}(V) = \ell^n\, \overline{\mu}_{-\infty,1}(V).$$

[25] **[E]** The limiting variable is r, as in the statement and the limit that follows. Both source versions switch to a different variable here.

Proof. It suffices to show that $\overline{\mu}_{-\infty,2}(V) \geqslant \ell^n\,\overline{\mu}_{-\infty,1}(V)$ under the hypothesis $\liminf \varphi_2/\varphi_1 > 1$. Fix $r < R_2$ and set $\varphi = \max(\varphi_1 - A, \varphi_2)$, where A is chosen sufficiently large that φ agrees with φ_2 near $S_2(r)$. Let μ_r be the measures associated with φ. The hypothesis $\liminf \varphi_2/\varphi_1 > 1$ implies that there is a $t < r$ such that φ agrees with $\varphi_1 - A$ on $B_1(t) = \{\varphi_1 < t\}$. Thus

$$\overline{\mu}_{-\infty,1}(V) = \lim_{t\to-\infty} \overline{\mu}_{t,1}(V) = \lim_{t\to-\infty} \overline{\mu}_t(V) \leqslant \mu_r(V) = \mu_{r,2}(V),$$

and hence

$$\overline{\mu}_{-\infty,1}(V) \leqslant \lim_{r\to-\infty} \overline{\mu}_{r,2}(V) = \overline{\mu}_{-\infty,2}(V). \qquad \square$$

Under the preceding hypotheses, one may conjecture that the inequality of measures
$\overline{\mu}_{-\infty,2} \geqslant \ell^n\,\overline{\mu}_{-\infty,1}$ always holds, but Theorem 4.2 does not enable us to prove it. We do, however, have the following special result:

Corollary 4.3. – *With the notation of Theorem* 4.2*, let $A \subset S_1(-\infty)$ be a Borel set that is a union of connected components of $S_1(-\infty)$, and let $\mathbf{1}_A$ be the characteristic function of A. Then, for every psh function $V \geqslant 0$,*

$$\overline{\mu}_{-\infty,2}(\mathbf{1}_A V) \geqslant \ell^n\,\overline{\mu}_{-\infty,1}(\mathbf{1}_A V).$$

In particular, if $S_1(-\infty)$ is totally disconnected, then $\overline{\mu}_{-\infty,2} \geqslant \ell^n\,\overline{\mu}_{-\infty,1}$.

Proof. There is an increasing sequence of compact sets $K_\nu \subset A$ such that $\overline{\mu}_{-\infty,j}(A \smallsetminus K_\nu) < 2^{-\nu}$, $j = 1, 2$. The equivalence relation whose classes are the connected components of $S_1(-\infty)$ has a closed graph (since $S_1(-\infty)$ is compact). The saturation $\tilde{K}_\nu$ of K_ν is therefore a compact subset of A; moreover, $\tilde{K}_\nu$ is the intersection of a decreasing sequence of subsets that are both open and closed in $S_1(-\infty)$ (cf. Bourbaki [Bo], chap. II, §4, no. 4). We may therefore assume that A is open and closed in $S_1(-\infty)$. There is then an open set $U \Subset X$ such that $A = U \cap S_1(-\infty)$ and $\partial U \cap S_1(-\infty) = \emptyset$. Let $r_0 = \inf_{\partial U} \varphi_1 > -\infty$ and

$$\Omega = \big\{z \in U\,;\ \varphi_1(z) < r_0\big\}.$$

The open set Ω is a union of connected components of $B_1(r_0)$, and hence Ω is Stein; moreover, $\varphi_1 : \Omega \to [-\infty, r_0[$ is exhaustive. Set

$$\varphi_\nu = \max\big(\varphi_2, \nu(\varphi_1 - r_0 + 1)\big).$$

For $\nu > \sup_\Omega \varphi_2$, the map $\varphi_\nu : \Omega \to [-\infty, \nu[$ is exhaustive, while for $\nu \geqslant \ell$ we have

$$\liminf_{\varphi_1(z)\to-\infty} \frac{\varphi_\nu(z)}{\varphi_1(z)} = \liminf \frac{\varphi_2}{\varphi_1} = \ell.$$

By Theorem 4.2 applied to φ_1 and φ_ν on Ω, we obtain

$$\overline{\mu}_{-\infty,\nu}(\mathbf{1}_\Omega V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(\mathbf{1}_\Omega V).$$

If ℓ is > 0 (the only case of interest), then $S_2(-\infty) \supset S_1(-\infty)$, and therefore $S_\nu(-\infty) = S_1(-\infty)$ and $\Omega \cap S_1(-\infty) = U \cap S_1(-\infty) = A$. Consequently,

$$(dd^c\varphi_\nu)^n(\mathbf{1}_A V) = \overline{\mu}_{-\infty,\nu}(\mathbf{1}_A V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(\mathbf{1}_A V).$$

Now observe that the sequence φ_ν decreases to φ_2 on the open set $B_1(r_0-1)$ as $\nu \to +\infty$, so that $(dd^c\varphi_\nu)^n$ converges weakly to $(dd^c\varphi_2)^n$ on $B_1(r_0-1)$ by 2.6 (a) and 2.10. Since A is compact, $A \subset S_1(-\infty) \subset B_1(r_0-1)$, and $\mathbf{1}_A V$ is upper semicontinuous, we conclude in the limit that

$$\overline{\mu}_{-\infty,2}(\mathbf{1}_A V) = (dd^c\varphi_2)^n(\mathbf{1}_A V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(\mathbf{1}_A V). \qquad \square$$

In the classical computation that follows, we shall need to evaluate the mass $\|\mu_{-\infty}\|$ from the function $\varphi^* = e^\varphi$. For this purpose, Proposition 3.10 gives $\mu_r^* = \mu_{\log r}$, and hence

$$\text{(4.4)} \qquad \overline{\mu}_{-\infty}(1) = \lim_{r\to 0} r^{-n}\mu_r^*(1) = \lim_{r\to 0} r^{-n}\int_{\{\varphi^*<r\}} (dd^c\varphi^*)^n.$$

Proposition 4.5. – *Let $\varphi = \log\varphi^*$ be a continuous psh function on $\mathbb{C}^n$, where φ^* is homogeneous of degree $\ell > 0$ and $(\varphi^*)^{-1}(0) = \{0\}$. Then* [26]

$$(dd^c\varphi)^n = (2\pi\ell)^n\delta_0$$

where δ_0 is the Dirac measure at 0.

Proof. The homogeneity of φ^* implies $(dd^c\varphi)^n = 0$ on $\mathbb{C}^n \smallsetminus \{0\}$. In the special case $\varphi^*(z) = |z|^2$, one finds $(dd^c\varphi^*)^n = 4^n\, n!\, d\lambda$ ($d\lambda$ = Lebesgue measure), and hence $\overline{\mu}_{-\infty}(1) = (4\pi)^n$ and $(dd^c\varphi)^n = \overline{\mu}_{-\infty} = (4\pi)^n\delta_0$. The general case follows from Theorem 4.2. $\square$

Example 4.6. – To illustrate the preceding results, consider the case where $\varphi(z) = \log\max(|z_1|,\dots,|z_n|)$ on $\mathbb{C}^n$. Since φ depends on only $n-1$ variables near every point in the complement of the "diagonal" $\Delta = \{|z_1| = \cdots = |z_n|\}$, the homogeneity of e^φ implies that $(dd^c\varphi)^{n-1} = 0$ on $\mathbb{C}^n \smallsetminus \Delta$. Proposition 3.3[27] then shows that the measure μ_r is supported on the distinguished boundary

$$\Gamma(r) = \big\{|z_1| = \cdots = |z_n| = e^r\big\} = S(r) \cap \Delta$$

[26] **[E]** The zero set is $\{0\}$. The equality printed in the source versions is incorrect.

[27] **[E]** The result used here is Proposition 3.3. Both source versions cite a nonexistent Corollary 3.7.

of the polydisc $B(r)$. Since μ_r is invariant under the rotations preserving $B(r)$, and since $\|\mu_r\| = (2\pi)^n$ by 4.5, it follows that $\mu_r = d\theta_1 \wedge \ldots \wedge d\theta_n$, where $z_j = e^{r+i\theta_j}$, $1 \leqslant j \leqslant n$. We also obtain:

$$(dd^c\varphi)^n = (2\pi)^n \delta_0 \,,$$
$$(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi = dr \wedge d\theta_1 \wedge \ldots \wedge d\theta_n \quad \text{on } \Delta \smallsetminus \{0\}. \qquad \square$$

We now return to the general case. Let $x \in X$ be any point and let $w = (w_1, w_2, \ldots, w_N)$ be the N coordinate functions associated with an embedding of a neighborhood $U \subset X$ of x into $\mathbb{C}^N$, with $w(x) = 0$. There are a neighborhood $\Omega \Subset U$ of x and $r_0 < 0$ such that the function $\varphi_1(z) = \log |w(z)|^2 : \Omega \to \,]-\infty, r_0[$ is exhaustive. Formula 4.4 then gives (cf. [De5]):

$$\overline{\mu}_{-\infty,1}(1) = (4\pi)^n \, \nu([X], x)$$

where $\nu([X], x)$ is the Lelong number at x of the current of integration over X in $\mathbb{C}^N$, equal, by P. Thie [Th], to the algebraic multiplicity $m(X, x)$ of X at x. We conclude that

$$\overline{\mu}_{-\infty,1} = (4\pi)^n \, m(X, x) \, \delta_x \,. \tag{4.7}$$

For an arbitrary function φ, we obtain the following result, which is well known at least when X is smooth.

Corollary 4.8. – *Denote by $\nu(\varphi, x)$ the Lelong number of φ at any point $x \in X$. Then*

$$(dd^c\varphi)^n \geqslant \overline{\mu}_{-\infty} \geqslant (2\pi)^n \sum_{x \in X} m(X, x) \, \nu(\varphi, x)^n \, \delta_x \,.$$

Proof. With the preceding notation, one of the equivalent definitions of the Lelong number is

$$\nu(\varphi, x) = \liminf_{z \to x} \frac{\varphi(z)}{\log |w(z)|}.$$

Set $\varphi_1(z) = \chi(z) \log |w(z)|^2 + A\psi(z)$, where χ is $\mathcal{C}^\infty$ with compact support in Ω, $\chi \equiv 1$ near x, ψ is strictly psh of class $\mathcal{C}^2$ on X, and $A > 0$ is sufficiently large. By Corollary 4.3 and formula (4.7), we have

$$\liminf_{z \to x} \frac{\varphi(z)}{\varphi_1(z)} = \frac{1}{2} \nu(\varphi, x),$$

and hence

$$\overline{\mu}_{-\infty} \geqslant \Big(\frac{1}{2} \nu(\varphi, x) \Big)^n \overline{\mu}_{-\infty,1} \geqslant (2\pi)^n \, m(X, x) \, \nu(\varphi, x)^n \, \delta_x. \qquad \square$$

5 Maximum principle.

Let φ be a continuous psh exhaustion function on a complex space X. We shall see that plurisubharmonic functions on X satisfy the maximum principle with respect to the Monge-Ampère measures associated with φ.

Theorem 5.1. – *If $B(r) = \{\varphi < r\} \neq \emptyset$, then $\|\mu_r\| > 0$, and for every psh function V on X one has:*[28]

$$\sup_{B(r)} V = \textit{essential supremum of } V \textit{ with respect to } \mu_r.$$

Example 4.6 shows that the plurisubharmonicity assumption on V in Theorem 5.1 is relevant.

Proof. There is no loss of generality in assuming that $V \geqslant 0$.[29] We shall then prove that $\sup_{B(r)} V = \|V\|_{L^\infty(\mu_r)}$ by applying Jensen's formula to a suitably chosen exhaustion function φ'.

Let ψ be a strictly psh function of class $\mathcal{C}^2$ on X, let $z_0 \in B(r) \cap X_{\text{reg}}$ be a regular point, and let $U \Subset B(r) \cap X_{\text{reg}}$ be a neighborhood of z_0. For $\varepsilon > 0$ sufficiently small, the function

$$\varphi'(z) = \max\left(\varphi(z), \varphi(z_0), r - \sqrt{\varepsilon} + \varepsilon\psi(z)\right)$$

is equal to $\varepsilon\psi(z) + \text{Const.}$ on U and coincides with φ in a neighborhood of $S(r)$. The measure μ_r may therefore equally well be defined using φ', which gives

$$\begin{aligned}\mu_r(V) &= \int_{-\infty}^{r} dt \int_{B(r)\cap\{\varphi'<t\}} dd^cV \wedge (dd^c\varphi')^{n-1} + \int_{B(r)} V\,(dd^c\varphi')^n \\ &\geqslant \varepsilon^n \int_U V\,(dd^c\psi)^n.\end{aligned}$$

In particular, $\|\mu_r\| = \mu_r(1) > 0$. Now replace V by V^p and let p tend to $+\infty$. We obtain:

$$V(z_0) \leqslant \lim_{p\to+\infty}\left[\int_U V^p\,(dd^c\psi)^n\right]^{1/p} \leqslant \lim_{p\to+\infty}\left[\varepsilon^{-n}\mu_r(V^p)\right]^{1/p} = \|V\|_{L^\infty(\mu_r)}.$$

Consequently, we obtain

$$\sup_{B(r)} V = \sup_{B(r)\cap X_{\text{reg}}} V \leqslant \|V\|_{L^\infty(\mu_r)}.$$

[28] **[P]** The published statement ends with $\sup_{B(r)} V$, which is the quantity being characterized.

[29] **[P]** The proof begins by replacing V by a nonnegative function, so the correct assumption is $V \geqslant 0$. The signs in the two source versions are inconsistent.

In the other direction, the inequality

$$\|V\|_{L^\infty(\mu_r)} \leqslant \sup_{S(r)} V$$

is obvious. Thus, if we prove the left-continuity of the function $r \mapsto \|V\|_{L^\infty(\mu_r)}$, we will have

$$\|V\|_{L^\infty(\mu_r)} \leqslant \lim_{t<r,\, t\to r} \sup_{S(t)} V \leqslant \sup_{B(r)} V.$$

Lemma 5.2. – *For every psh function $V \geqslant 0$, the map $r \mapsto \|V\|_{L^\infty(\mu_r)}$ is increasing and left-continuous.*

Proof. Formula 3.4 shows that the function $r \mapsto \mu_r(V)$ is increasing and left-continuous. On every interval $]-\infty, r_0]$, $r_0 < R$, the functions

$$r \mapsto \left[\|\mu_{r_0}\|^{-1}\mu_r(V^p)\right]^{1/p}$$

are therefore increasing and left-continuous, and form an increasing family with respect to p by Hölder's inequality (the measure $\|\mu_{r_0}\|^{-1}\mu_r$ has mass $\leqslant 1$). The limit as $p \to +\infty$, namely $r \mapsto \|V\|_{L^\infty(\mu_r)}$, is therefore increasing and left-continuous on $]-\infty, r_0]$. $\square$

6 Convexity properties of psh functions.

A well-known result of P. Lelong (cf. [Le1]) states that the supremum, the mean, and more generally the L^p mean of a psh function on the Euclidean sphere of radius r in $\mathbb{C}^n$ are convex functions of $\log r$. We propose to extend these properties to a much more general setting.

Let X be a Stein space of pure dimension n, and let $\varphi : X \to [-\infty, R[$ be a continuous psh exhaustion function. We assume that φ is homogeneous Monge-Ampère, i.e. that there exists $A \in\,]-\infty, R[$ such that

$$(6.1) \qquad (dd^c\varphi)^n = 0 \quad \text{on the open set } \{\varphi > A\}.$$

For every psh function V on X and every $r > A$, Theorem 3.4 then shows that the left derivative

$$(6.2) \qquad \frac{d}{dr_-}\,\mu_r(V) = \int_{B(r)} dd^cV \wedge \alpha^{n-1}$$

is positive and increasing in r, whence the following

Theorem 6.3. – *The mean-value function $r \mapsto M_V(r) = \mu_r(V)$ is increasing and convex on $]A, R[$.*

The classical case mentioned at the outset corresponds to the ball of radius e^R in $\mathbb{C}^n$ with $\varphi(z) = \log|z|$, $A = -\infty$. More generally, we have a convexity result for the L^p means defined by

$$M_V^p(r) = \Big[\mu_r(V_+^p)\Big]^{1/p}, \qquad p \in [1, +\infty[\,.$$

Theorem 6.4. – *The function $r \mapsto M_V^p(r)$ is increasing and convex on $]A, R[$.*

Proof. By regularization, we reduce to the case where V is a psh function > 0 of class $\mathcal{C}^\infty$. Given $\varepsilon > 0$, consider the function[30]

$$h_\varepsilon(r) = \int_{r-\varepsilon}^{r} \mu_t(V^p)\,dt = \int_{B(r)\smallsetminus B(r-\varepsilon)} V^p\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi, \qquad r \in\,]A+\varepsilon, R[$$

(the last equality follows from Proposition 3.9 (a)). Since $\mu_r(V^p) = \lim_{\varepsilon\to 0} \varepsilon^{-1} h_\varepsilon(r)$,[31] it suffices to prove that $h_\varepsilon^{1/p}$ is convex for every $\varepsilon > 0$. We must therefore verify

[30] **[P]** The published formula includes the integration factor dt and the full annular form $\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi$.

[31] **[P]** The factor ε^{-1} is required when recovering $\mu_r(V^p)$ from the integral over the interval $[r-\varepsilon, r]$.

the inequality[32]

$$h_\varepsilon h_\varepsilon'' - \Big(1-\frac{1}{p}\Big)h_\varepsilon'^2 \geqslant 0 \tag{6.5}$$

where the second derivative h_ε'' is, say, computed from the left. By Proposition 3.9 (b) and assumption (6.1), we obtain

$$\begin{aligned} h_\varepsilon'(r) &= \mu_r(V^p) - \mu_{r-\varepsilon}(V^p) \\ &= \int_{B(r)\smallsetminus B(r-\varepsilon)} d\big[V^p\,\alpha^{n-1}\wedge d^c\varphi\big] \\ &= \int_{B(r)\smallsetminus B(r-\varepsilon)} p\,V^{p-1}\,dV\wedge\alpha^{n-1}\wedge d^c\varphi. \end{aligned}$$

Formula (6.2) also implies

$$h_\varepsilon''(r) = \int_{B(r)\smallsetminus B(r-\varepsilon)} dd^c(V^p)\wedge\alpha^{n-1}.$$

By the Cauchy-Schwarz inequality, we obtain

$$h_\varepsilon'(r)^2 \leqslant \int_{B(r)\smallsetminus B(r-\varepsilon)} V^p\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi\ \cdot \int_{B(r)\smallsetminus B(r-\varepsilon)} p^2 V^{p-2}\,dV\wedge d^cV\wedge\alpha^{n-1},$$

and the desired relation (6.5) follows from the inequality[33]

$$dd^c(V^p) \geqslant p(p-1)\,V^{p-2}dV\wedge d^cV. \qquad \square$$

Corollary 6.6. – *The functions defined by*

(a) $M_V^{\exp}(r) = \log\mu_r(e^V)$,

(b) $M_V^{\infty}(r) = \sup_{B(r)} V$,

are increasing and convex on $]A,R[$.

Proof. Property (a) follows from Theorem 6.4 and the equality

$$\log\mu_r(e^V) = \lim_{p\to+\infty} p\bigg\{\Big[\mu_r\Big(1+\frac{V}{p}\Big)_+^p\Big]^{1/p} - 1\bigg\}.$$

On the other hand, the maximum principle (Theorem 5.1) yields

$$\sup_{B(r)} V = \lim_{\lambda\to+\infty}\frac{1}{\lambda}\log\mu_r(e^{\lambda V})$$

[32] **[P]** The published paper numbers this convexity inequality as (6.5). The number is restored because it is used later.

[33] **[P]** The coefficient in this Cauchy–Schwarz estimate follows the published calculation; without it, the deduction of (6.5) is incorrect.

hence (b) follows from (a). $\square$

With a view toward applications to the study of fiber spaces, we now prove a version of Theorem 6.4 with parameters. Let $\pi : X \to Y$ be a morphism of analytic spaces of pure dimensions $\dim X = m+n$, $\dim Y = m$, and let $\varphi : X \to [-\infty, +\infty[$ be continuous psh, let $R : Y \to\,]-\infty, +\infty]$ be lower semicontinuous, and let $A : Y \to [-\infty, +\infty[$ be upper semicontinuous, satisfying the properties below.

Assumptions 6.7. –

(a) *π is surjective, and the fibers $\pi^{-1}(y)$, $y \in Y$ are of pure dimension n.*

(b) *π is a Stein morphism, i.e. Y has an open covering $(\Omega_j)_{j\in J}$ such that $\pi^{-1}(\Omega_j)$ is Stein for every $j \in J$.*

(c) *$\varphi(x) < R(\pi(x))$ and $A(y) < R(y)$ for all $x \in X$, $y \in Y$.*

(d) *For every $y \in Y$ and every $r < R(y)$,*[34] *there exists a neighborhood U of y in Y such that $\pi^{-1}(U) \cap B(r) \Subset X$.*

(e) *$(dd^c\varphi)^n \equiv 0$ on the open set $\{x \in X\,;\ \varphi(x) > A(\pi(x))\}$.*

We again write $B(r) = \{\varphi < r\}$, $S(r) = \{\varphi = r\}$, and $\alpha = dd^c\varphi$. Under assumptions (c) and (d), §3 associates with each fiber $\pi^{-1}(y)$ a family of measures $\mu_{y,r}$ supported on $\pi^{-1}(y) \cap S(r)$ for $r \in\,]-\infty, R(y)[$. Given a psh function V on X, we introduce the mean values

$$\begin{aligned}
M_V(y,r) &= \mu_{y,r}(V),\\
M_V^p(y,r) &= \big[\mu_{y,r}(V_+^p)\big]^{1/p} \quad \text{if } p \in [1, +\infty[,\\
M_V^{\exp}(y,r) &= \log \mu_{y,r}(e^V),\\
M_V^{\infty}(y,r) &= \sup_{\pi^{-1}(y)\cap B(r)} V.
\end{aligned}$$

Proposition 6.8. – *For every fixed r, the maps $y \mapsto \mu_{y,r}(V)$ and $y \mapsto M_V^p(y,r)$ are weakly psh in the sense of Definition* 1.9 *on the open set $\{y \in Y\,;\ A(y) < r < R(y)\}$.*

Proof. Since the result is local on Y, assumption 6.7 (b) allows us to assume that X and Y are Stein. Passing to a decreasing limit then reduces us to the case where V is psh of class $\mathcal{C}^\infty$; if $p > 1$, we may further assume that $V > 0$.

[34] **[P]** Assumption 6.7(d) concerns each radius $r < R(y)$. The variable r is missing from the web TeX.

Let $\varepsilon > 0$ be arbitrary, and let $\chi :]r-\varepsilon, r[\to \mathbb{R}$ be a nonzero $\mathcal{C}^\infty$ function with compact support such that $\chi \geqslant 0$. In analogy with Theorem 6.4, introduce the auxiliary function[35]

$$h(y) = \int_{r-\varepsilon}^{r} \mu_{y,t}(V^p)\,\chi(t)\,dt = \int_{\pi^{-1}(y)} V^p\,\chi(\varphi)\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi$$

defined on the open set

$$U_\varepsilon = \{y \in Y\,;\ A(y) + \varepsilon < r < R(y)\}.$$

To conclude, it suffices to show that $h^{1/p}$ is weakly psh on U_ε. If $p > 1$, we must therefore show that

$$h\,dd^c h - \Big(1 - \frac{1}{p}\Big) dh \wedge d^c h \geqslant 0.$$

Let u, v, and w be real differential forms of class $\mathcal{C}^\infty$ on Y, compactly supported in U_ε, of respective bidegrees (m,m), $(m,m-1) \oplus (m-1,m)$, and $(m-1,m-1)$. By Fubini's theorem, which we first apply under the assumption that φ is of class $\mathcal{C}^\infty$, we obtain

$$\int_Y hu = \int_X V^p\,\chi(\varphi)\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi \wedge \pi^* u;$$

the case where φ is merely continuous follows by taking a decreasing limit (Theorem 2.6). We now observe that the integrand is supported in

$$\pi^{-1}(\operatorname{Supp} u) \cap \big(B(r) \smallsetminus B(r-\varepsilon)\big) \Subset X$$

(assumption 6.7 (d)) and that the current $\chi(\varphi)\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi$ is d-closed (assumption 6.7 (e)). By an integration by parts on Y and another one in the reverse direction on X, we successively obtain

$$\int_Y dh \wedge v = \int_X d(V^p) \wedge \chi(\varphi)\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi \wedge \pi^* v, \tag{6.9}$$

$$\int_Y dd^c h \wedge w = \int_X dd^c(V^p) \wedge \chi(\varphi)\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi \wedge \pi^* w. \tag{6.10}$$

Assume that the $(m-1,m-1)$-form w is $\geqslant 0$. Equality (6.10) already shows that $\int_Y dd^c h \wedge w \geqslant 0$, hence $dd^c h \geqslant 0$ on U_ε, which settles the case $p = 1$. In the general case $p > 1$, let γ be a real $\mathcal{C}^\infty$ 1-form on Y, and set $\gamma^c = i(\gamma^{0,1} - \gamma^{1,0})$.

[35] **[P/E]** In this definition of h, the scalar factors have been grouped correctly and the annular measure includes the full wedge product.

Equality (6.9), combined with the Cauchy-Schwarz inequality, yields[36]

$$\begin{aligned}\int_Y dh\wedge\gamma^c\wedge w &= \int_X p\,V^{p-1}dV\wedge\pi^*\gamma^c\wedge\chi(\varphi)\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi\wedge\pi^*w\\ &\leqslant \frac{1}{2}\int_X \big(V^p\,\pi^*(\gamma\wedge\gamma^c)+p^2\,V^{p-2}dV\wedge d^cV\big)\wedge\chi(\varphi)\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi\wedge\pi^*w\\ &\leqslant \frac{1}{2}\int_Y \Big(h\,\gamma\wedge\gamma^c+\frac{p}{p-1}dd^ch\Big)\wedge w,\end{aligned}$$

since $dd^cV^p \geqslant p(p-1)\,V^{p-2}dV\wedge d^cV$. Since this holds for every form $w\geqslant 0$, we deduce the following inequality in the sense of currents:

$$dh\wedge\gamma^c+\gamma\wedge d^ch\leqslant h\,\gamma\wedge\gamma^c+\frac{p}{p-1}\,dd^ch.$$

Observe that h is everywhere >0 on U_ε by 4.1; letting γ tend to dh/h, we obtain the desired inequality

$$\frac{1}{h}\,dh\wedge d^ch\leqslant\frac{p}{p-1}dd^ch.$$

To see that h is locally bounded above on Y, it suffices to consider the case where $V\equiv 1$. Equality (6.9) then shows that $dh=0$, so h is locally constant on Y_{reg}.[37] □

Proposition 6.8 in fact contains the following more general result, which was our main objective.

Theorem 6.11. – *The functions on $Y\times\mathbb{C}$ defined by*[38]

$$(y,z)\mapsto M_V(y,\operatorname{Re}z),\quad M_V^p(y,\operatorname{Re}z),\quad M_V^{\exp}(y,\operatorname{Re}z),\quad M_V^\infty(y,\operatorname{Re}z)$$

are weakly psh on the open set

$$\big\{(y,z)\in Y\times\mathbb{C}\,;\ A(y)<\operatorname{Re}z<R(y)\big\}.$$

Proof. Consider the morphism

$$\tilde\pi=\pi\times\mathrm{Id}:X\times\mathbb{C}\to Y\times\mathbb{C}$$

and equip $X\times\mathbb{C}$, $Y\times\mathbb{C}$ with the functions

$$\tilde\varphi(x,z)=\varphi(x)-\operatorname{Re}z,\quad \tilde R(y,z)=R(y)-\operatorname{Re}z,\quad \tilde A(y,z)=A(y)-\operatorname{Re}z,$$

[36] **[P/E]** The displayed estimate includes the factors and parentheses needed for the Cauchy–Schwarz argument that follows.

[37] **[E]** Here the regular locus is Y_{reg}. When $V=1$, the identity $dh=0$ shows that the resulting function h is locally constant there.

[38] **[E]** All four mean-value functions depend on the same base variable y. The source versions change this variable inconsistently.

so that assumptions 6.7 (a-e) are satisfied with respect to these data. If $\tilde V(x,z) = V(x)$, then by construction

$$\tilde\mu_{(y,z),0}(\tilde V) = \mu_{y,\operatorname{Re} z}(V),$$

and Theorem 6.11 therefore follows from Proposition 6.8. $\square$

Corollary 6.12. – *Let $(X_j)_{1\leqslant j\leqslant k}$ be Stein spaces of pure dimension n_j, and let $\varphi_j : X_j \to [-\infty, R_j[$ be continuous psh exhaustion functions such that $(dd^c\varphi_j)^{n_j} \equiv 0$ on the open set $\{x \in X_j\,;\ \varphi_j(x) > A_j\}$. If V is psh on $X_1 \times \cdots \times X_k$, the functions*

$$M_V(r_1,\ldots,r_k) = \mu_{1,r_1} \otimes \cdots \otimes \mu_{k,r_k}(V),$$

$$M_V^p(r_1,\ldots,r_k) = M_{V_+^p}(r_1,\ldots,r_k)^{1/p},$$

$$M_V^{\exp}(r_1,\ldots,r_k) = \log M_{e^V}(r_1,\ldots,r_k),$$

$$M_V^\infty(r_1,\ldots,r_k) = \sup_{B(r_1)\times\cdots\times B(r_k)} V$$

are jointly convex in the variables $(r_1,\ldots,r_k) \in \prod_{1\leqslant j\leqslant k}]A_j, R_j[$, and increasing with respect to each r_j.

More generally, if X_0 is an analytic space of pure dimension n and if V is psh on $X_0 \times X_1 \times \cdots \times X_k$, the function[39]

$$M_V^p(x_0, \operatorname{Re} z_1,\ldots,\operatorname{Re} z_k) = M_{V(x_0,\bullet)}^p(\operatorname{Re} z_1,\ldots,\operatorname{Re} z_k)$$

(resp. $p = \emptyset$, $\exp$, ∞) is psh on the open set

$$X_0 \times \prod_{1\leqslant j\leqslant k} \{A_j < \operatorname{Re} z_j < R_j\} \subset X_0 \times \mathbb{C}^k.$$

Proof. It suffices to prove the last assertion. We argue by induction on k. For $k=1$, Theorem 6.11, applied to $\pi : X = X_0 \times X_1 \to X_0 = Y$ and $\varphi = \varphi_1$, shows that the function

$$(x_0, z_1) \mapsto M_V^p(x_0, \operatorname{Re} z_1)$$

is weakly psh. If, in addition, V is continuous, this function is separately continuous in x_0 and convex in $\operatorname{Re} z$, hence continuous in $(x_0, \operatorname{Re} z_1)$. By Corollary 1.12, $M_V^p(x_0, \operatorname{Re} z)$ is therefore psh. The case of arbitrary V follows by writing V as a decreasing limit of continuous psh functions.

For $k > 1$, the assertion for k follows from the cases 1 and $k-1$ by setting

$$W(x_0, x_1, \ldots, x_{k-1}, z_k) = M_{V(x_0,\ldots,x_{k-1},\bullet)}^p(\operatorname{Re} z_k)$$

[39] **[P]** The parameter is x_0, in agreement with the product $X_0 \times X_1 \times \cdots \times X_k$.

and observing that

$$M_V^p(x_0, \operatorname{Re} z_1, \ldots, \operatorname{Re} z_k) = M^p_{W(x_0,\bullet,z_k)}(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_{k-1}). \qquad \square$$

We conclude this section by reconsidering, in light of the preceding results, P. Lelong's convexity inequality, which precisely measures the variations in the growth of a psh function on a fiber space along the different fibers. This inequality was used by H. Skoda [Sk2] to construct the first counterexample to the problem, posed by J.-P. Serre in 1953, of determining whether a fiber space with Stein base and Stein fiber is itself Stein; see also [De1], [De2] for other counterexamples and [De3] for a simple and rapid construction.

Let Ω be an irreducible Stein space of dimension m, which will serve as the base of the fiber space, and let X be a Stein space of pure dimension n, which will be the fiber. Assume that there exist continuous psh exhaustion functions $\psi : \Omega \to [-\infty, R[$, $\varphi : X \to [-\infty, +\infty[$ such that

$$(dd^c\psi)^m = 0 \quad \text{on} \quad \{\psi > A\}, \qquad (dd^c\varphi)^n = 0 \quad \text{on } \{\varphi > 0\}.$$

For example, if X is an affine algebraic variety of dimension n, there exists a finite morphism $F : X \to \mathbb{C}^n$ (the Noether normalization theorem), and it suffices to take $\varphi(z) = \log \|F(z)\|$, where $\|\ \ \|$ is a norm on $\mathbb{C}^n$; the same argument applies locally on Ω to establish the existence of ψ.

Let V be a psh function on $\Omega \times X$, and let a, b, c, r be real numbers such that $A < a < b < c < R$ and $r > 0$. The convexity property in Corollary 6.12 shows that

$$M_V^\infty(b, r) \leqslant M_V^\infty(a, \sigma r) + \Big(1 - \frac{1}{\sigma}\Big) \Big[M_V^\infty(c, 0) - M_V^\infty(a, \sigma r)\Big]$$

with $\sigma = \frac{c-a}{c-b}$. It follows from Theorem 7.5, proved in the next section, that $M_V^\infty(a, r) \to +\infty$ as $r \to +\infty$, provided that V is nonconstant on at least one fiber $\{z\} \times X$, $z \in B(a)$. There then exists a constant r_0 depending on a, b, c, V such that

$$M_V^\infty(b, r) < M_V^\infty(a, \sigma r) \quad \text{for } r > r_0, \text{ where } \ \sigma = \frac{c-a}{c-b}. \tag{6.13}$$

If ω is a relatively compact open subset of Ω, we now set

$$M_V^\infty(\omega\,;\, r) = \sup_{\omega \times B(r)} V.$$

By an elementary compactness and connectedness argument (cf. [Le2], Theorem 6.5.4), we then deduce from (6.13) the following result:

Corollary 6.14 (P. Lelong's inequality)**.** – *Let Ω be an irreducible complex space, let ω_1, ω_2 be two relatively compact open subsets of Ω, and let V be a psh* *function on $\Omega \times X$, assumed to be nonconstant on at least one fiber $\{z\} \times X$. Then there exists a constant $\sigma > 1$ depending only on $\omega_1, \omega_2, \Omega$, and a constant r_0 depending additionally on V, such that for every $r > r_0$ one has*

$$M_V^\infty(\omega_2\,;\,r) < M_V^\infty(\omega_1\,;\,\sigma r).$$

In practical applications, one faces the problem of explicitly computing the constant σ. Inequality (6.13) provides a complete theoretical answer to this problem. If $\omega_1, \omega_2 \Subset \Omega$ are open subsets of $\mathbb{C}$, one seeks a harmonic function ψ on $\Omega \smallsetminus \overline{\omega}_1$ that tends to 0 on $\partial\omega_1$ and to 1 on $\partial\Omega$ (resp. to $+\infty$ if $\partial\Omega$ has capacity 0); one extends ψ by 0 on ω_1 and sets $b = \sup_{\omega_2}\psi$. Any constant $\sigma > \frac{1}{1-b}$ (resp. $\sigma > 1$) then answers the question. The case where the base Ω has dimension $m > 1$ amounts to solving an analogous Dirichlet problem for the Monge-Ampère equation $(dd^c\psi)^m = 0$ on $\Omega \smallsetminus \overline{\omega}_1$. It is easy to see that the constant σ thus obtained is the best possible. Indeed, an elementary calculation shows that the function

$$\chi(t,r) = \exp\Big(\frac{r}{1-t} + \frac{1}{(1-t)^2}\Big)$$

is increasing and convex on $[0,1[\,\times\,[0,+\infty[$. The psh function $V = \chi(\psi_+, \varphi_+)$ then contradicts (6.13) for every $\sigma \leqslant \frac{1}{1-b}$.

7 Growth at infinity of psh functions.

Let X be an irreducible Stein space of dimension n, and let $\varphi : X \to [-\infty, +\infty[$ be a continuous psh exhaustion function (thus, here $R = +\infty$). We write

$$\tau(r) = \|\mu_r\| = \int_{B(r)} \alpha^n$$

where $\alpha = dd^c\varphi$; this is the volume of the pseudoball $B(r) = \{\varphi < r\}$. Jensen's formula 3.4 then relates the growth of the current dd^cV to the growth of V. More precisely:

Proposition 7.1. – *Let V be a psh function on X, $r_0 \in \mathbb{R}$, and $\varepsilon \in\,]0,1[$. There exists a constant $C > 0$ depending on V, ε, r_0 such that, for every $r \geqslant r_0$, one has*[40]

$$(1-\varepsilon)r \int_{B(r_0)} dd^cV \wedge \alpha^{n-1} \leqslant \mu_r(V_+) + C\,\tau(r).$$

[40] **[P]** The published estimate contains the factor $1-\varepsilon$ and integrates over the fixed pseudoball $B(r_0)$.

Proof. Set $V_\nu = \max(V, -\nu)$, $\nu \in \mathbb{N}$. The measures $dd^cV_\nu \wedge \alpha^{n-1}$ converge weakly to $dd^cV \wedge \alpha^{n-1}$ as $\nu \to +\infty$; hence

$$\liminf_{\nu\to+\infty} \int_{B(r_0)} dd^cV_\nu \wedge \alpha^{n-1} \geqslant \int_{B(r_0)} dd^cV \wedge \alpha^{n-1}.$$

There therefore exists $\nu \in \mathbb{N}$ (depending on V, ε, r_0) such that

$$\int_{B(r_0)} dd^cV_\nu \wedge \alpha^{n-1} \geqslant (1-\varepsilon) \int_{B(r_0)} dd^cV \wedge \alpha^{n-1}.$$

On the other hand, Formula 3.4 applied to V_ν gives[41]

$$(r - r_0) \int_{B(r_0)} dd^cV_\nu \wedge \alpha^{n-1} \leqslant \mu_r(V_\nu) - \int_{B(r)} V_\nu\, \alpha^n \leqslant \mu_r(V_+) + \nu\, \tau(r).$$

Combining these two inequalities yields Proposition 7.1 with

$$C = \nu + \frac{(1-\varepsilon)r_0}{\tau(r_0)} \int_{B(r_0)} dd^cV \wedge \alpha^{n-1}. \qquad \square$$

Throughout the remainder of this section, we impose a moderate-growth assumption on the volume of X.

Assumption 7.2. – $\displaystyle\lim_{r\to+\infty} \frac{\tau(r)}{r} = 0.$

We then obtain the following fundamental inequality as an immediate consequence of Proposition 7.1:

Corollary 7.3. – *Under assumption* (7.2), *one has, for every psh function V:*

$$\int_X dd^cV \wedge \alpha^{n-1} \leqslant \liminf_{r\to+\infty} \frac{1}{r}\mu_r(V_+).$$

This inequality will be used mainly through the following lemma:

Lemma 7.4. – *Let ψ be a strictly psh function of class $\mathcal{C}^2$ on X, and let $r_1 < r_2$ with $B(r_2) \neq \emptyset$. Then there exists a constant $C(r_1, r_2) > 0$ such that, for every psh function V, one has:*

$$\int_{B(r_1)} dd^cV \wedge (dd^c\psi)^{n-1} \leqslant C(r_1, r_2) \int_{B(r_2)} dd^cV \wedge \alpha^{n-1}.$$

[41] **[P]** The second comparison is an upper bound, as required to derive Proposition 7.1.

Proof. Set $\varphi' = \max(\varphi, r_1 + \varepsilon\psi + \sqrt{\varepsilon})$, where $\varepsilon > 0$ is chosen sufficiently small that $\varphi' = \varphi$ in a neighborhood of $S(r_2)$ and $\varphi' = r_1 + \varepsilon\psi + \sqrt{\varepsilon}$ on $B(r_1)$. Stokes' theorem gives[42]

$$\int_{B(r_2)} dd^cV\wedge(dd^c\varphi)^{n-1} = \int_{B(r_2)} dd^cV\wedge(dd^c\varphi')^{n-1} \geqslant \varepsilon^{n-1}\int_{B(r_1)} dd^cV\wedge(dd^c\psi)^{n-1}.$$

□

Theorem 7.5. – *Every psh function V on X satisfying one of the growth assumptions below is constant:*

(a) $\liminf_{r\to+\infty} \dfrac{1}{r}\mu_r(V_+) = 0.$

(b) $\sup_{B(r)} V = o\Big(\dfrac{r}{\tau(r)}\Big)$ *as* $r \to +\infty$.

Proof. Theorem 5.1 gives

$$\mu_r(V_+) \leqslant \|\mu_r\| \sup_{B(r)} V_+ = \tau(r) \sup_{B(r)} V_+,$$

so assumption 7.5 (b) implies 7.5 (a). Under assumption 7.5 (a), Corollary 7.3 and Lemma 7.4 show that $dd^cV = 0$, i.e. V is pluriharmonic. For every $x \in X$, the function $z \mapsto \tilde V(z) = \max(V(z), V(x))$ still satisfies 7.5 (a), and is therefore pluriharmonic. By the maximum principle, $-\tilde V$ is constant on X (since X is assumed irreducible), i.e. $V \leqslant V(x)$; thus V is constant. □

In the usual setting of the Hermitian space $\mathbb{C}^n$ and the exhaustion function $\varphi(z) = \log|z|$, Theorem 7.5 recovers (with a slightly simpler proof) a result due to N. Sibony and P.-M. Wong. Define the logarithmic order $\rho(V)$ of a psh function V in $\mathbb{C}^n$ (resp. of an entire function F) by

$$1 + \rho(V) = \limsup_{r\to+\infty} \frac{\log \sup_{|z|<r} V(z)}{\log\log r}, \qquad (\text{resp.}\ \ \rho(F) = \rho(\log|F|).$$

In other words, V and F have logarithmic order $\leqslant \rho$ if, for every $\varepsilon > 0$, one has

$$V_+(z) \leqslant \varphi(z)^{1+\rho+\varepsilon}, \qquad (\text{resp.}\ \ |F(z)| \leqslant \exp\big(\varphi(z)^{1+\rho+\varepsilon}\big))$$

when $|z|$ is sufficiently large. In particular, every polynomial has logarithmic order zero, and every entire function of finite logarithmic order has order zero in the usual sense.

Corollary 7.6 ([SW])**.** – *Let F be a nonconstant entire function of logarithmic order $\rho < 1$, and let X be an irreducible component of the hypersurface $F^{-1}(0)$.*

[42] **[P]** The last integral is over $B(r_1)$, precisely the region on which $\varphi' = r_1 + \varepsilon\psi + \sqrt{\varepsilon}$.

Then every psh function V on X of logarithmic order $< 1-\rho$ is constant. In particular, bounded psh and holomorphic functions on X are constant.

Proof. Theorem 7.5 reduces the problem to estimating the volume of X with respect to φ, which is a classical problem. Indeed, the current of integration over X is bounded above by $\frac{1}{2\pi}dd^c\log|F|$ by the Lelong-Poincaré equation (cf. [Le1]); hence

$$\tau(r)=\int_{X\cap\{\varphi<r\}}(dd^c\varphi)^{n-1}\leqslant\int_{\{\varphi<r\}}\frac{1}{2\pi}dd^c\log|F|\wedge\alpha^{n-1}.$$

After a possible translation, we may assume that $F(0)\neq 0$. Proposition 4.5 and Formula 3.4, applied to $V=\frac{1}{2\pi}\log|F|$ in $\mathbb{C}^n$, then give

$$\int_{-\infty}^{r}\tau(t)\,dt\leqslant\mu_r\Big(\frac{1}{2\pi}\log|F|\Big)-(2\pi)^n\frac{1}{2\pi}\log|F(0)|\leqslant C\,r^{1+\rho+\varepsilon},\qquad r\geq r_0(\varepsilon).$$

Since the function τ is increasing, it follows that

$$\tau(r)\leqslant\frac{1}{r}\int_{r}^{2r}\tau(t)\,dt\leqslant C'\,r^{\rho+\varepsilon}.$$

The conclusion now follows from 7.5 (b). □

8 Holomorphic φ-polynomial functions.

We retain the notation and assumptions of §7: X denotes an irreducible Stein space of dimension n, equipped with a psh exhaustion function φ satisfying the volume-growth condition (7.2).

Definition 8.1. – *If F is a holomorphic function on X, the degree of F with respect to φ is the number*[43]

$$\delta_\varphi(F)=\limsup_{r\to+\infty}\ \frac{1}{r}\,\mu_r(\log_+|F|)\in[0,+\infty].$$

The obvious inequalities $\log_+|FG|\leqslant\log_+|F|+\log_+|G|$ and[44]

$$\log_+|\lambda F+\mu G|\leqslant\log_+|F|+\log_+|G|+\log_+(|\lambda|+|\mu|),$$

[43] **[P]** The definition in the published paper contains the factor $1/r$. Without it, the quantity would not be the degree used below.

[44] **[E]** The constant in the sum estimate is $\log_+(|\lambda|+|\mu|)$. The corresponding expression in both source versions is malformed.

together with (7.2), imply, for all scalars $\lambda, \mu \in \mathbb{C}$:

$$\delta_\varphi(FG) \leqslant \delta_\varphi(F) + \delta_\varphi(G), \qquad \delta_\varphi(\lambda F + \mu G) \leqslant \delta_\varphi(F) + \delta_\varphi(G).$$

Thus the set of holomorphic functions of finite degree is an integral $\mathbb{C}$-algebra.

Notation 8.2. – *We denote by:*

(a) $A_\varphi(X)$ *the algebra of holomorphic functions of finite degree, which will be called* φ*-polynomial functions.*

(b) $K_\varphi(X)$ *the field of quotients* F/G *with* $F, G \in A_\varphi(X)$*, called the field of* φ*-rational functions.*

The terminology is justified by Theorem 8.5 below. In all examples known to us, the equality $A_\varphi(X) = K_\varphi(X) \cap \mathcal{O}(X)$ holds, but we do not know whether this property holds in general. On the other hand, if X is normal, $A_\varphi(X)$ is an integrally closed subalgebra of $K_\varphi(X)$ (an immediate verification).

With these definitions, we have the following fundamental inequality, which follows from Corollary 7.3[45] applied to $V = \log|F|$.

Proposition 8.3. – *Let* $[Z_F] = \frac{1}{2\pi} dd^c \log|F|$ *be the zero divisor of a function* $F \in A_\varphi(X)$ *that is not identically zero. Then*

$$2\pi \int_X [Z_F] \wedge \alpha^{n-1} \leqslant \delta_\varphi(F). \qquad \square$$

Corollary 8.4. – *Let* a *be a regular point of* X*. Denote by* $\mathrm{ord}_a(F)$ *the order of vanishing of a holomorphic function* F *at* a*. There exists a constant* $C(a) > 0$ *such that, for every nonzero function* $F \in A_\varphi(X)$*, one has:*

$$\mathrm{ord}_a(F) \leqslant C(a)\, \delta_\varphi(F).$$

Proof. Let $(z_1, z_2, \ldots, z_n)$ be a system of local coordinates on X, centered at a, such that the ball $|z| \leqslant \varepsilon$ is relatively compact in X. Lemma 7.4 implies the existence of a constant $C_1 > 0$ such that

$$\int_{|z| \leqslant \varepsilon} [Z_F] \wedge (dd^c|z|^2)^{n-1} \leqslant C_1 \int_X [Z_F] \wedge \alpha^{n-1}.$$

Corollary 8.4 then follows from Proposition 8.3 and the classical inequality of P. Lelong [Le1]:

$$\frac{1}{(4\pi\varepsilon^2)^{n-1}} \int_{|z| \leqslant \varepsilon} [Z_F] \wedge (dd^c|z|^2)^{n-1} \geqslant \mathrm{ord}_a(F). \qquad \square$$

[45] **[E]** The estimate follows from Corollary 7.3. The cross-reference in the source versions points to the wrong result.

Using a classical argument going back to Poincaré and developed by Siegel [Si1], [Si2], we shall now deduce a very general algebraicity theorem.

Theorem 8.5. – *The transcendence degree over $\mathbb{C}$ of the field $K_\varphi(X)$ of φ-rational functions satisfies:*

(a) $0 \leqslant \deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X) \leqslant n = \dim X$.

(b) *If* $\deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X) = n$*, then* $K_\varphi(X)$ *is a finitely generated extension of* $\mathbb{C}$.

Proof. Let $F_1, \ldots, F_N$ be φ-polynomial functions, let $(k_1, \ldots, k_N)$ be an N-tuple of integers $\geqslant 0$, and let $P \in \mathbb{C}[X_1, \ldots, X_N]$ be a polynomial in N variables such that $\deg_{X_j} P \leqslant k_j$ and $P(F_1, \ldots, F_N) \not\equiv 0$. Then

$$\log_+ |P(F_1, \ldots, F_N)| \leqslant \sum_{1\leqslant j\leqslant N} k_j \log_+ |F_j| + \text{Const.},$$
$$\delta_\varphi(P(F_1, \ldots, F_N)) \leqslant \sum_{1\leqslant j\leqslant N} k_j\, \delta_\varphi(F_j).$$

Corollary 8.4 therefore gives the inequality

$$\operatorname{ord}_a P(F_1, \ldots, F_N) \leqslant C(a) \sum_{1\leqslant j\leqslant N} k_j\, \delta_\varphi(F_j). \tag{8.6}$$

Assume that $F_1, \ldots, F_N$ are algebraically independent. Then the dimension of the vector space of polynomials $P(F_1, \ldots, F_N)$ is $(k_1 + 1) \ldots (k_N + 1)$. For every integer $s \geqslant 0$ and every given point $a \in X_{\text{reg}}$, the homogeneous linear system

$$\frac{\partial^\nu}{\partial z^\nu} P(F_1, \ldots, F_N)_{|z=a} = 0, \qquad \nu \in \mathbb{N}^n, \quad |\nu| \leqslant s,$$

has a nonzero solution as soon as this dimension exceeds the number of equations, which is $\binom{n+s}{n} \leqslant \frac{1}{n!}(n+s)^n$. The function $P(F_1, \ldots, F_N)$ then vanishes to order at least $s+1$ at the point a, and choosing s such that $s \leqslant C(a) \sum k_j\, \delta_\varphi(F_j) < s+1$ contradicts inequality (8.6), unless

$$(k_1 + 1) \ldots (k_N + 1) \leqslant \binom{n+s}{s} \leqslant \frac{1}{n!}\Big[n + C(a) \sum_{1\leqslant j\leqslant N} k_j\, \delta_\varphi(F_j)\Big]^n.$$

Now take $k_1 = \cdots = k_N = k$ and let k tend to $+\infty$. The preceding inequality shows that $(k+1)^N \leqslant \text{Const.}(k+1)^n$; hence $N \leqslant n$, proving property (a).

Now assume that $\deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X) = n$, and let $F_1, \ldots, F_n$ be n algebraically independent functions in $A_\varphi(X)$. To prove (b), it suffices to bound the degree of the algebraic extension $[K_\varphi(X) : \mathbb{C}(F_1, \ldots, F_n)]$.[46] If $F_{n+1} \in A_\varphi(X)$ is algebraic of

[46] **[P]** A closing bracket is missing from this extension degree in the web TeX; it has been restored.

degree d over $\mathbb{C}(F_1,\ldots,F_n)$, the monomials $F_1^{\ell_1}\ldots F_n^{\ell_n}F_{n+1}^{\ell_{n+1}}$ are linearly independent whenever $\ell_{n+1}<d$. The preceding argument, applied with $k_{n+1}=d-1$,[47] therefore gives

$$(k_1+1)\ldots(k_n+1)d\leqslant\frac{1}{n!}\Big[n+C(a)\sum_{1\leqslant j\leqslant n}k_j\,\delta_\varphi(F_j)+(d-1)\,\delta_\varphi(F_{n+1})\Big]^n.$$

Take $k_1\sim q_1k,\ldots,k_n\sim q_nk$, where $q_1,\ldots,q_n$ are real numbers >0 and $k\to+\infty$. In the limit, we obtain

$$q_1\ldots q_nd\leqslant\frac{1}{n!}\Big[C(a)\sum_{1\leqslant j\leqslant n}q_j\,\delta_\varphi(F_j)+(d-1)\,\delta_\varphi(F_{n+1})\Big]^n,$$

and the choice $q_j=1/\delta_\varphi(F_j)$ gives the desired explicit upper bound for the degree:

$$d\leqslant\frac{(nC(a))^n}{n!}\;\delta_\varphi(F_1)\cdots\delta_\varphi(F_n).\qquad\square$$

We point out that Theorem 8.5 (b) says nothing about the algebra $A_\varphi(X)$ itself; as we shall see in §10, it may very well happen that the algebra $A_\varphi(X)$ is *not finitely generated.*

As an application, consider the special case where X is a (closed) analytic subset of pure dimension n in $\mathbb{C}^N$, equipped with the usual exhaustion function $\varphi(z)=\log(1+|z|^2)$. The associated metric $\alpha=dd^c\varphi$ is identified with the Fubini-Study metric on projective space $\mathbb{P}^N$, while the metric $\beta=dd^ce^\varphi$ coincides with the flat Hermitian metric on $\mathbb{C}^N$. Proposition 3.10 implies the relations[48]

$$\int_{X\cap\{|z|<r\}}\beta^n=(1+r^2)^n\int_{X\cap\{\varphi<\log(1+r^2)\}}\alpha^n,$$
$$\mathrm{Vol}_\alpha(X)=\int_X\alpha^n=\lim_{r\to+\infty}\frac{1}{(1+r^2)^n}\int_{X\cap\{|z|<r\}}\beta^n.$$

Theorem 8.5 then gives an elementary proof of the following classical result due to W. Stoll [St1].

Corollary 8.7. – *Let X be a closed analytic subset of pure dimension n in $\mathbb{C}^N$, whose projective volume* $\mathrm{Vol}_\alpha(X)$ *is finite, i.e. whose Euclidean volume satisfies the estimate*

$$\mathrm{Vol}_\beta(X\cap\{|z|<r\})\leqslant C\cdot r^{2n},\qquad C\geqslant 0.$$

Then X is algebraic.

[47] **[P]** The relevant index is k_{n+1}, corresponding to the algebraic element F_{n+1}.
[48] **[E]** The limiting Euclidean-volume formula requires the exact factor $(1+r^2)^{-n}$.

Proof. Every irreducible component of X has volume at least equal to the volume of an n-plane (cf. [Le1]), so there are only finitely many such components, and we may assume that X is irreducible.

We now observe that the polynomials $P \in \mathbb{C}[z_1, \ldots, z_N]$ induce φ-polynomial functions on X in the sense of Definitions 8.1 and 8.2. Indeed, the obvious estimate $\log_+ |P| \leqslant \frac{1}{2} \deg(P)\, \varphi + \mathrm{Const.}$ yields

$$\delta_\varphi(P) = \limsup_{r \to +\infty} \frac{1}{r}\, \mu_r(\log_+ |P|) \leqslant \frac{1}{2} \mathrm{Vol}_\alpha(X) \cdot \deg(P).$$

Now consider the restriction morphism

$$\mathbb{C}[z_1, \ldots, z_N] \to A_\varphi(X)$$

and let I be its kernel. Since $A_\varphi(X)$ is an integral domain, I is a prime ideal; moreover, by definition, the irreducible algebraic zero set $V(I)$ contains X. Theorem 8.5 (a) shows that $\mathbb{C}[z_1, \ldots, z_N]/I \subset A_\varphi(X)$ has transcendence degree at most $n = \dim X$; consequently, $\dim V(I) \leqslant n$ and $X = V(I)$. □

Remark 8.8. – In the situation of the corollary, there is an isomorphism

$$\mathbb{C}[z_1, \ldots, z_N]/I \xrightarrow{\simeq} A_\varphi(X),$$

in particular, $A_\varphi(X)$ is finitely generated. Otherwise, there would exist a finitely generated algebra B such that $\mathbb{C}[z_1, \ldots, z_N]/I \subsetneq B \subset A_\varphi(X)$. Let $M = \mathrm{Spm}\, B$ be the affine algebraic variety associated with B (see [Di], vol. 2, chap. I, for the basic formalism concerning algebraic varieties); the preceding inclusions would then induce, respectively, an algebraic morphism $M \to V(I)$ and an analytic morphism $V(I) = X \to M$, inverse to each other. Consequently, the morphism $V(I) \to M$ would be algebraic, and we would have $\mathbb{C}[z_1, \ldots, z_N]/I = B$, contrary to the assumption. □

We now show how these results carry over to the case of “polynomial” sections of a line bundle. Let L be a Hermitian line bundle over X, let D be the canonical Hermitian connection on L, and let $c(L) = D^2$ be the curvature $(1,1)$-form of L.

If σ is a nonzero holomorphic section of L, then, for every $\varepsilon > 0$, we obtain:[49]

$$i\partial\overline{\partial} \log(\varepsilon + |\sigma|^2) = i\partial \left[\frac{\langle \sigma, D\sigma \rangle}{\varepsilon + |\sigma|^2} \right] = \frac{\varepsilon \langle D\sigma, D\sigma \rangle}{(\varepsilon + |\sigma|^2)^2} - \frac{|\sigma|^2}{\varepsilon + |\sigma|^2}\, ic(L).$$

[49] **[E]** The curvature term in this identity is $ic(L)$. The notation in the source versions is inconsistent with the preceding definition of $c(L)$.

Jensen's formula 3.4 applied to the function $V = \frac{1}{2}\log(\varepsilon + |\sigma|^2)$ then gives, taking into account that $V \geqslant \log \varepsilon^{1/2}$:

$$\begin{aligned}\int_0^r dt \int_{B(t)} & \frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon + |\sigma|^2)^2} \wedge \alpha^{n-1} \\ &\leqslant \int_0^r dt \int_{B(t)} \frac{|\sigma|^2}{\varepsilon + |\sigma|^2}\, ic(L) \wedge \alpha^{n-1} + \mu_r(V) - \mu_0(V) - \int_{B(r)\smallsetminus B(0)} V\,\alpha^n \\ &\leqslant r \int_X \big[ic(L) \wedge \alpha^{n-1}\big]_+ + \mu_r(V) + \tau(r)\,\log \varepsilon^{-\frac{1}{2}},\end{aligned}$$

where $[ic(L) \wedge \alpha^{n-1}]_+$ denotes the positive part of the measure $ic(L) \wedge \alpha^{n-1}$. Divide this inequality by r and let r tend to $+\infty$. Since $V \leqslant \log_+ |\sigma| + \frac{1}{2}\log(1+\varepsilon)$, we obtain

$$\int_X \frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon + |\sigma|^2)^2} \wedge \alpha^{n-1} \leqslant \limsup_{r\to+\infty} \frac{1}{r}\,\mu_r(\log_+ |\sigma|) + \int_X \big[ic(L) \wedge \alpha^{n-1}\big]_+ .$$

As ε tends to 0, the term $\frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon+|\sigma|^2)^2}$ converges weakly to the integration current $2\pi[Z_\sigma]$ associated with the zero divisor of σ. We therefore obtain the following generalization of inequality 8.3.

Proposition 8.9. – *For every holomorphic section $\sigma \not\equiv 0$ of L, one has*

$$2\pi \int_X [Z_\sigma] \wedge \alpha^{n-1} \leqslant \delta_\varphi(\sigma) + \delta_\varphi(L)$$

where $\delta_\varphi(\sigma)$, $\delta_\varphi(L)$ denote the respective "degrees" of σ and L:

$$\begin{aligned}\delta_\varphi(\sigma) &= \limsup_{r\to+\infty} \frac{1}{r}\mu_r(\log_+ |\sigma|),\\ \delta_\varphi(L) &= \int_X \big[ic(L) \wedge \alpha^{n-1}\big]_+ .\end{aligned}$$

The algebraicity theorem can now be stated as follows.

Theorem 8.10. – *Denote by $K_\varphi(X, L)$ the field of meromorphic functions on X of the form σ_1/σ_2 with $\sigma_j \in H^0(X, \mathcal{O}(L^m))$, $m \in \mathbb{N}$, $\delta_\varphi(\sigma_j) < +\infty$, $j = 1, 2$. If the bundle L has finite degree $\delta_\varphi(L)$, then:*

(a) $0 \leqslant \deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X, L) \leqslant n = \dim X$;

(b) *If $\deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X, L) = n$, the field $K_\varphi(X, L)$ is finitely generated.*

Proof. Let $F_1 = \sigma_1'/\sigma_1, \ldots, F_N = \sigma_N'/\sigma_N$ be elements of $K_\varphi(X, L)$ with $\sigma_j, \sigma_j' \in H^0(X, \mathcal{O}(L^{m_j}))$,[50] and let $P \in \mathbb{C}[X_1, \ldots, X_N]$ be a polynomial such that $\deg_{X_j} P \leqslant$

[50] **[E]** Both the numerator and denominator are global sections of L^{m_j}, so they both lie in $H^0(X, \mathcal{O}(L^{m_j}))$.

k_j. Set

$$\sigma = P\Big(\frac{\sigma'_1}{\sigma_1},\ldots,\frac{\sigma'_N}{\sigma_N}\Big)\sigma_1^{k_1}\cdots\sigma_N^{k_N} \in H^0(X,\mathcal{O}(L^m)),\qquad m=\sum k_j m_j.$$

Inequality (8.6) then generalizes as follows:

$$\mathrm{ord}_a(\sigma) \leqslant C(a)\sum_{1\leqslant j\leqslant N} k_j\Big[\max(\delta_\varphi(\sigma_j),\delta_\varphi(\sigma'_j)) + m_j\delta_\varphi(L)\Big], \tag{8.11}$$

and the rest of the proof is identical to that of 8.5. $\square$

B. Geometric characterization of affine algebraic varieties

9 Statement of the algebraicity criterion.

The purpose of the following sections is to show that affine algebraic varieties are characterized among Stein spaces by simple geometric conditions, namely finiteness of the Monge-Ampère volume and a suitable lower bound on the Ricci curvature.

Recall that an affine algebraic variety is, by definition, a closed algebraic subvariety of some $\mathbb{C}^N$. In the case of a space X with isolated singularities, we obtain the following necessary and sufficient characterization.

Theorem 9.1. – *Let X be a complex analytic space of dimension n, with only finitely many singular points. Then X is analytically isomorphic to an affine algebraic variety if and only if X admits a strictly psh exhaustion function φ of class $\mathcal{C}^\infty$ having the properties* (a), (b), (c) *below.*

(a) $\mathrm{Vol}(X) = \int_X (dd^c\varphi)^n < +\infty$;

(b) *The Ricci curvature of the metric $\beta = dd^c(e^\varphi)$ admits a lower bound of the form*
$$\mathrm{Ricci}(\beta) \geqslant -\frac{1}{2} dd^c\psi,$$
with $\psi \in L^1_{\mathrm{loc}}(X, \mathbb{R}) \cap \mathcal{C}^0(X_{\mathrm{reg}}, \mathbb{R})$ and $\psi \leqslant A\varphi + B$, where A, B are constants $\geqslant 0$;

(c) *φ has only finitely many critical points on X_{reg}.*

If these conditions are satisfied, the ring $R_\varphi(X) = K_\varphi(X) \cap \mathcal{O}(X)$ (cf. Definition 8.2) is a finitely generated $\mathbb{C}$-algebra of transcendence degree n. The algebraic structure X_{alg} is then defined as the unique algebraic structure on X whose ring of regular functions is $R_\varphi(X)$.

Extending this characterization to analytic spaces with arbitrary singularities presents difficulties that will be examined in §14.

The roles of the various assumptions in Theorem 9.1 are divided, roughly speaking, as follows. The existence of a strictly psh exhaustion function φ guarantees that X is a Stein variety, by H. Grauert's solution of the Levi problem [Gr].

Under assumption (a), Theorem 8.5 further implies that the field of φ-rational functions has finite transcendence degree. Assumption (b), in turn, ensures the existence of sufficiently many φ-polynomial functions, thanks to the L^2 estimates of Hörmander-Nakano-Bombieri-Skoda for the $\overline{\partial}$ operator. We point out that condition (b) cannot be replaced by a condition on the curvature of the metric $dd^c\varphi$ (cf. Remark 10.2). On the other hand, one obtains an equivalent condition by replacing β with any metric γ such that

$$\exp(-A_1\varphi - B_1) \leqslant \gamma \leqslant \exp(A_2\varphi + B_2),$$

for example, the metric $\gamma = dd^c \log(1 + e^\varphi)$ or $\gamma = dd^c(\varphi^2)$.

Finally, assumption (c) implies, by Morse theory, that X has the same homotopy type as a finite cell complex, and hence that the cohomology of X is of finite type. We do not in fact know whether assumption (c) is really indispensable once X is assumed irreducible. Without assumption (c), one can already show that X is the union of an increasing sequence of quasi-affine algebraic varieties (= Zariski-open subsets of affine varieties), cf. Proposition 13.1. This result yields the following improvement of Theorem 9.1.

Theorem 9.1′. – *Theorem* 9.1 *remains valid if assumptions* (a,b,c) *are weakened as follows:*

(a') = (a) : $\mathrm{Vol}(X) = \int_X (dd^c\varphi)^n < +\infty$;

(b') $\mathrm{Ricci}(\beta) \geqslant -\frac{1}{2}dd^c\psi$, *where* $\psi \in L^1_{\mathrm{loc}}(X,\mathbb{R}) \cap \mathcal{C}^0(X_{\mathrm{reg}},\mathbb{R})$ *satisfies an estimate of the form*

$$\int_X \exp(c\psi - A\varphi)\,\beta^n < +\infty, \qquad c > 0, \quad A > 0;$$

(c') *the even-degree cohomology spaces* $H^{2q}(X_{\mathrm{reg}};\mathbb{R})$ *are finite-dimensional.*

Assumptions (a'), (b') still imply that $X = \bigcup_{k\in\mathbb{N}} X_k$ with X_k quasi-affine, $X_k \subset X_{k+1}$, and assumption (c') implies that the sequence X_k is necessarily stationary; consequently, X is algebraic. Observe that assumption (c') is always satisfied if $n = 1$; when $n = 2$ or $n = 3$, it is equivalent merely to assuming $\dim H^2(X;\mathbb{R}) < +\infty$, since the groups $H^q(X;\mathbb{R})$ always vanish for $q > n$ when X is Stein.

The plausibility of Theorems 9.1 and 9.1' was suggested to us in part by the work of W. Stoll and D. Burns on parabolic varieties. We recall here the fundamental result of W. Stoll (1980), which characterizes strictly parabolic varieties of arbitrary radius.

Theorem 9.2 (cf. [St2] and [Bu])**. –** *Let M be a connected complex analytic manifold of dimension n. Assume that there exist a real number* $R \in \,]0,+\infty]$

and a strictly psh exhaustion function $\tau : M \to [0, R^2[$ *of class* $\mathcal{C}^\infty$*, such that* $\log \tau$ *is psh and satisfies* $(dd^c \log \tau)^n \equiv 0$ *on* $M \smallsetminus \tau^{-1}(0)$*. Then there exists a biholomorphic map* $F : B(R) \to M$ *from the ball of radius* R *in* $\mathbb{C}^n$ *such that* $F^*\tau(z) = |z|^2$*.*

If the strict plurisubharmonicity assumption on τ is dropped, it is easy to see that every affine algebraic variety M still satisfies the condition of Theorem 9.2 (with $R = +\infty$): it suffices to choose $\tau(z) = |\pi(z)|^2$, where $\pi : M \to \mathbb{C}^n$ is a finite proper morphism. This observation led D. Burns to pose the problem of characterizing such varieties in terms of exhaustion functions with special properties. We mention, in particular, the following open problems.

Problem 9.3. – *Consider a Stein manifold* M *of dimension* n*, admitting a psh exhaustion function* $\tau : M \to [0, +\infty[$ *of class* $\mathcal{C}^\infty$ *such that* $\log \tau$ *is psh and satisfies* $(dd^c \log \tau)^n \equiv 0$ *on* $M \smallsetminus \tau^{-1}(0)$*. Is the manifold* M *affine algebraic?*

Problem 9.4. – *Characterize the manifolds* M *admitting a strictly psh exhaustion function* $\tau : M \to [0, +\infty[$ *such that* $\log \tau$ *is psh and satisfies* $(dd^c \log \tau)^n \equiv 0$ *outside a compact set.*

D. Burns showed that there exist affine algebraic varieties that do not satisfy condition 9.4: this is the case for $M = (\mathbb{C}^*)^n$, $n \geq 2$. Nevertheless, condition 9.4 is satisfied by a "generic" affine variety, namely a variety embedded in $\mathbb{C}^N$ whose projective completion is smooth and transverse to the hyperplane at infinity.

Shortly after proving Theorem 9.1, we learned that N. Mok had previously obtained a geometric condition sufficient (though not necessary in general) for a manifold to be affine algebraic.

Theorem 9.5 ([Mok 1,2,3]). – *Let* X *be a complete Kähler manifold of dimension* n*, with positive bisectional curvature. Fix* $x_0 \in X$*, let* $B(x_0, r)$ *be the geodesic ball, and let* $d(x_0, x)$ *denote the distance. Suppose that there are constants* $c, C > 0$ *such that*

(a) $\mathrm{Vol}(B(x_0, r)) \geqslant c\, r^{2n}$,

(b) $0 <$ *scalar curvature* $\leqslant C/d(x_0, x)^2$.

Then X *is biholomorphic to an affine algebraic variety.*

N. Mok deduced from this theorem that every surface X of positive Riemannian curvature satisfying assumptions 9.5 (a), (b) is in fact isomorphic to $\mathbb{C}^2$. The analogous result in dimension $n > 2$ remains a conjecture. Theorem 9.5 relies essentially on work [MSY] of Mok, Siu, and Yau on solving the Poincaré-Lelong equation on Kähler manifolds with bisectional curvature > 0. Apart from this result, N. Mok's proof follows, in broad outline, an approach substantially parallel to ours.

The assumption of positive bisectional curvature nevertheless appears rather restrictive and does not, in general, cover the case of affine algebraic varieties (the Euclidean curvature of such a variety is always negative, cf. §10). We mention, however, some results known when the curvatures are not necessarily positive. Siu and Yau [SY] proved that a simply connected complete Kähler manifold X whose sectional curvature satisfies

$$-\frac{C}{d(x_0,x)^{2+\varepsilon}} < \text{sectional curvature} < 0$$

is biholomorphic to $\mathbb{C}^n$. The same holds if the curvature satisfies[51]

$$\big|\text{sectional curvature}\big| \leqslant \frac{C_\varepsilon}{d(x_0,x)^{2+\varepsilon}}$$

with a sufficiently small constant C_ε (cf. [MSY]).

10 Necessity of the volume and curvature conditions.

We shall prove here that conditions (a), (b), (c) of Theorem 9.1 are satisfied by every irreducible algebraic subvariety $X \subset \mathbb{C}^N$ of dimension n.

In this case, choose $\varphi(z) = \log(1+|z|^2)$, so that the metric $\alpha = dd^c\varphi$ coincides with the Fubini-Study metric on projective space $\mathbb{P}^N$. Since the closure $\overline{X}$ of X in $\mathbb{P}^N$ is a compact algebraic subvariety, we obtain

$$\int_X (dd^c\varphi)^n = \int_{\overline{X}} \alpha^n < +\infty,$$

and consequently condition (a) is satisfied.

By the Bertini–Sard theorem, the set of critical values of φ on X_{reg} is finite. Hence the critical set of φ is compact. By slightly perturbing φ in $\mathcal{C}^\infty(X,\mathbb{R})$ in a neighborhood of this compact set ([Mi], Cor. 6.8), we construct a function φ' whose critical points are nondegenerate. The critical points of φ' are then finite in number [assumption (c)].

It remains to show that X satisfies curvature condition 9.1 (b) with respect to the metric

$$\beta = dd^c(e^\varphi) = dd^c|z|^2 = 2i\sum_{j=1}^N dz_j \wedge d\overline{z}_j\,,$$

[51] **[P]** The published estimate places absolute-value bars around the sectional curvature. They are necessary because the estimate controls its magnitude.

which we shall verify by an explicit calculation of $\mathrm{Ricci}(\beta_{|X})$ and ψ.

Let $(P_1, \ldots, P_m)$ be a system of polynomial generators for the ideal $I(X)$ of the subvariety X in $\mathbb{C}[z_1, \ldots, z_N]$, and let $s = \operatorname{codim} X = N - n$. For every pair of multi-indices

$$K = \{k_1 < \ldots < k_s\} \subset \{1, \ldots, m\}, \qquad L = \{\ell_1 < \ldots < \ell_s\} \subset \{1, \ldots, N\}$$

of length s, consider the partial Jacobian

$$J_{K,L}(z) = \det\Big(\partial P_{k_i}/\partial z_{\ell_j}\Big)_{1 \leqslant i,j \leqslant s},$$

and set

$$\psi(z) = \log\Bigg(\sum_{|K|=|L|=s} |J_{K,L}|^2\Bigg).$$

The functions $J_{K,L}$ are polynomials; in particular, there exist constants $A, B \geqslant 0$ such that $\psi \leqslant A\varphi + B$. The following proposition shows, moreover, that φ, ψ satisfy curvature inequality 9.1 (b).

Proposition 10.1. – *Denote by $U_K = U_{k_1,\ldots,k_s}$ the open subset of X on which the differentials $dP_{k_1}, \ldots, dP_{k_s}$ are linearly independent. Then:*

(a) $\mathrm{Ricci}(\beta_{|X}) = -\dfrac{1}{2}\, dd^c \log\Bigg(\displaystyle\sum_{|L|=s} |J_{K,L}|^2\Bigg)$ *on* U_K,

(b) $\mathrm{Ricci}(\beta_{|X}) \geqslant -\dfrac{1}{2}\, dd^c \log\Bigg(\displaystyle\sum_{|K|=|L|=s} |J_{K,L}|^2\Bigg)$ *on* X_{reg}.

Proof of (a). Let $x \in U_K$. Order the coordinates of $\mathbb{C}^N$ so that $(z_1, \ldots, z_n)$ is a system of local coordinates on X at the point x. In the Kähler case, the Ricci curvature of X is the negative of the curvature of the canonical bundle $\Lambda^n T^* X$. Thus, in a neighborhood of x, we have[52]

$$\mathrm{Ricci}(\beta_{|X}) = dd^c \log g,$$

where g is the norm, with respect to β, of the holomorphic $(n, 0)$-form $dz_1 \wedge \ldots \wedge dz_n$; g is given by

$$i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_n \wedge d\overline{z}_{n\,|X} = g^2\, \frac{1}{n!}\, \beta^n_{|X}.$$

[52] **[P]** The published formula is $\mathrm{Ricci}(\beta_{|X}) = dd^c \log g$. The expression $dd^c \log g^2$ in the web TeX is twice as large.

Let $L_0 = \{n+1, \dots, N\}$, let L be any multi-index of length $s = N - n$, and let $\complement L$ be its complement in $\{1, \dots, N\}$. If we set

$$dP_K = dP_{k_1} \wedge \dots \wedge dP_{k_s},$$

then, by the definition of $J_{K,L}$:

$$\begin{aligned} dP_K \wedge dz_{\complement L} &= \pm J_{K,L}\, dz_1 \wedge \dots \wedge dz_N, \\ i^{n^2+s^2} dP_K \wedge d\overline{P}_K \wedge dz_{\complement L} \wedge d\overline{z}_{\complement L} &= |J_{K,L}|^2 i\, dz_1 \wedge d\overline{z}_1 \wedge \dots \wedge i\, dz_N \wedge d\overline{z}_N \\ i^{s^2} dP_K \wedge d\overline{P}_K \wedge \frac{1}{n!}\,\beta^n &= 2^n \sum_{|L|=s} |J_{K,L}|^2\; i\, dz_1 \wedge d\overline{z}_1 \wedge \dots \wedge i\, dz_N \wedge d\overline{z}_N \\ &= 2^n |J_{K,L_0}|^{-2} \sum_{|L|=s} |J_{K,L}|^2\; i^{s^2}\, dP_K \wedge d\overline{P}_K \wedge i\, dz_1 \wedge d\overline{z}_1 \wedge \dots \wedge i\, dz_n \wedge d\overline{z}_n. \end{aligned}$$

"Canceling" $dP_K \wedge d\overline{P}_K$, we therefore find

$$g^2 = 2^{-n} |J_{K,L_0}|^2 \Big(\sum_{|L|=s} |J_{K,L}|^2 \Big)^{-1}$$

and since J_{K,L_0} is a holomorphic function invertible at x, formula (a) follows.

Proof of (b). Part (a) shows that the function

$$\log \Big(\sum_{|L|=s} |J_{K,L}|^2 \Big/ \sum_{|L|=s} |J_{K_0,L}|^2 \Big)$$

is pluriharmonic on the open set $U_K \cap U_{K_0}$. It is moreover locally bounded above, hence psh, on U_{K_0}. It follows that the function

$$\log \Big(\sum_{K,L} |J_{K,L}|^2 \Big/ \sum_{L} |J_{K_0,L}|^2 \Big)$$

is psh on U_{K_0}, and on this open set one therefore has:

$$dd^c \psi \geqslant dd^c \log \sum_L |J_{K_0,L}|^2 = -2\ \mathrm{Ricci}(\beta_{|X}). \qquad \square$$

Remark 10.2. – When X is a closed analytic subvariety of $\mathbb{C}^N$ and $\varphi(z) = \log(1+|z|^2)$, condition 9.1 (a), the finiteness of the volume, is by itself a sufficient condition for the algebraicity of X (W. Stoll's theorem, cf. Corollary 8.7). Nevertheless, the following example shows that, in general, the curvature condition 9.1 (b) cannot be dispensed with, even if 9.1' (c') is satisfied.

Choose $X = \mathbb{C} \smallsetminus E$, where $E = \{z_j\,;\ j \in \mathbb{N}\}$ is an infinite closed countable set, and set

$$\varphi(z) = \log(1 + |z|^2) - \sum_{j=0}^{+\infty} \varepsilon_j \log \frac{|z - z_j|}{1 + |z_j|}$$

where $\sum_{j=0}^{+\infty} \varepsilon_j = 1$ is a series with positive terms converging rapidly enough that φ belongs to $\mathcal{C}^\infty(\mathbb{C} \smallsetminus E)$. Since

$$\log \frac{|z - z_j|}{1 + |z_j|} \leqslant \log(1 + |z|),$$

we obtain

$$\varphi(z) \geqslant \log \frac{1 + |z|^2}{1 + |z|};$$

moreover, $\lim_{z \to z_j} \varphi(z) = +\infty$ for every $z_j \in E$. Thus φ is an exhaustion function on X. Furthermore, $dd^c\varphi = dd^c \log(1+|z|^2)$, hence $\int_X dd^c\varphi = 4\pi < +\infty$. However, X is not algebraic. □

This example also shows, incidentally, that condition 9.1 (b) cannot be replaced by a condition on the curvature of the metric $dd^c\varphi$.

It is also easy to see that the algebras $A_\varphi(X)$ and $R_\varphi(X) = K_\varphi(X) \cap \mathcal{O}(X)$ coincide with the algebra $\mathcal{A}_E$ of rational functions in $\mathbb{C}(z)$ whose poles lie in E. Indeed, every element $f \in \mathcal{A}_E$ clearly admits the upper bound $\log|f| \leqslant C_1\varphi + C_2$, hence $\mathcal{A}_E \subset A_\varphi(X) \subset R_\varphi(X)$; conversely, every element of $K_\varphi(X)$ is algebraic over $\mathbb{C}(z)$ by Theorem 8.5, so

$$R_\varphi(X) \subset \mathbb{C}(z) \cap \mathcal{O}(X) = \mathcal{A}_E.$$

It is clear that the algebra $\mathcal{A}_E$ is not finitely generated.

11 Existence of an embedding on an open subset of an algebraic variety.

The following sections 11–14 are devoted to proving the sufficiency of the algebraicity criterion 9.1'. We assume here that X is a smooth connected manifold, and that the given functions (φ, ψ) satisfy conditions 9.1'(a',b'). We also make the nonrestrictive additional assumption $\varphi \geqslant 0$. As before, set $\alpha = dd^c\varphi \geqslant 0$, $\beta = dd^c(e^\varphi)$; the measures μ_r are defined as in §3.

Definition 11.1. – *Let $p \in \,]0, +\infty]$. We denote by*

(a) $L^p_\varphi(X)$ *the vector space of measurable maps* $f : X \to \mathbb{C}$ *for which there exists a constant* $C \geqslant 0$ *such that:*

$$\int_X |f|^p e^{-C\varphi}\, \beta^n < +\infty, \qquad \textit{if } 0 < p < +\infty,$$

$$|f| \leqslant e^{C(1+\varphi)} \quad \textit{almost everywhere}, \qquad \textit{if } p = +\infty;$$

(b) $L^0_\varphi(X) = \bigcup_{p>0} L^p_\varphi(X)$;

(c) $A^p_\varphi(X) = L^p_\varphi(X) \cap \mathcal{O}(X)$, *if* $p \in [0, +\infty]$.

The usefulness of this definition appears in the two technical results below, which will be used repeatedly in what follows.

Lemma 11.2. – *The following properties hold:*

(a) $1 \in A^p_\varphi(X)$ *for every* $p > 0$;

(b) $L^p_\varphi(X) \subset L^q_\varphi(X)$ *and* $A^p_\varphi(X) \subset A^q_\varphi(X)$ *for all* $p \geqslant q > 0$;

(c) $L^0_\varphi(X)$ *is a* $\mathbb{C}$*-algebra* ;

(d) $A^0_\varphi(X)$ *is an integrally closed subalgebra of* $L^0_\varphi(X)$.

Lemma 11.3. – *One has the inclusion* $A^0_\varphi(X) \subset A_\varphi(X)$. *More precisely, if* $f \in A^0_\varphi(X)$ *satisfies*

$$\int_X |f|^p \exp(-C\varphi)\, \beta^n < +\infty,$$

then:

(a) $\delta_\varphi(f) \leqslant \dfrac{C-n}{p} \operatorname{Vol}(X)$;

(b) $\displaystyle\int_X |df|^p_\beta \exp\Big[\Big(\frac{p}{2} - C\Big)\varphi\Big]\, \beta^n < +\infty \quad$ *if* $p \in \,]0, 2]$,

where the norm $|df|_\beta$ *is computed with respect to the metric* β.

Proof of 11.2. Proposition 3.10 successively yields

$$v(r) := \int_{\{\varphi<r\}} \beta^n = e^{nr} \int_{\{\varphi<r\}} \alpha^n \leqslant e^{nr} \operatorname{Vol}(X),$$

$$\int_X e^{-(n+1)\varphi}\, \beta^n = \int_0^{+\infty} e^{-(n+1)r}\, dv(r) = (n+1) \int_0^{+\infty} e^{-(n+1)r}\, v(r)\, dr < +\infty,$$

which proves (a). Property (b) then follows from Hölder's inequality. The same inequality implies

$$\int_X |fg|^{\frac{pq}{p+q}} e^{-C\varphi}\,\beta^n \leqslant \left[\int_X |f|^p\, e^{-C\varphi}\,\beta^n\right]^{\frac{q}{p+q}} \left[\int_X |g|^q\, e^{-C\varphi}\,\beta^n\right]^{\frac{p}{p+q}}$$

for all $p, q > 0$. Thus we obtain the inclusion

$$L^p_\varphi(X) \cdot L^q_\varphi(X) \subset L_\varphi^{\frac{pq}{p+q}}(X), \tag{11.4}$$

and property (c) follows. We now verify assertion (d). Let f be a meromorphic function on X satisfying a monic equation over $A^0_\varphi(X)$ of the form

$$f^d + a_1 f^{d-1} + \cdots + a_{d-1} f + a_d = 0, \qquad a_j \in A^0_\varphi(X).$$

From this equation, we deduce the upper bound

$$|f| \leqslant 2 \max_{1\leqslant j\leqslant d} |a_j|^{1/j},$$

otherwise the equality

$$-1 = a_1 f^{-1} + \cdots + a_d f^{-d}$$

would lead, upon taking absolute values, to the absurd inequality

$$1 \leq 2^{-1} + \cdots + 2^{-d}.$$

Consequently, since X is smooth, f extends to a holomorphic function on X, and it is clear that $f \in A^0_\varphi(X)$. □

Proof of 11.3.

(a) One has $\beta^n \geqslant e^{n\varphi}(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi$; consequently (Proposition 3.9 (a)), the integral

$$\int_{-\infty}^{+\infty} e^{(n-C)r}\,\mu_r(|f|^p)\,dr = \int_X |f|^p e^{(n-C)\varphi}\,(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi \leqslant \int_X |f|^p e^{-C\varphi}\,\beta^n$$

is finite. Since the map $r \mapsto \mu_r(|f|^p)$ is increasing, we deduce that

$$\mu_r(|f|^p) \leqslant \exp\big((C-n)(r+1)\big) \int_r^{r+1} e^{(n-C)t}\,\mu_t(|f|^p)\,dt \leqslant C_1 e^{(C-n)r}$$

with a constant $C_1 \geqslant 0$. By Jensen's inequality and the inequality $\|\mu_r\| \leqslant \mathrm{Vol}(X)$, we obtain

$$\frac{\mu_r\big(\log(1+|f|^p)\big)}{\|\mu_r\|} \leqslant \log\left[\frac{\mu_r(1+|f|^p)}{\|\mu_r\|}\right] \leqslant (C-n)r + C_2\,,$$

$$\delta_\varphi(f) = \limsup_{r\to+\infty}\ \frac{1}{r}\,\mu_r(\log_+|f|) \leqslant \frac{C-n}{p}\ \mathrm{Vol}(X).$$

(b) To bound $|df|_\beta$, observe that

$$dd^c(|f|^p)\wedge\beta^{n-1}=\frac{p^2}{2n}\,|f|^{p-2}|df|_\beta^2\,\beta^n.$$

By Stokes' theorem, this equality implies, for every $r>0$:

$$\int_{B(r)}|f|^{p-2}|df|_\beta^2\Big(1-\frac{\varphi}{r}\Big)^2e^{(1-C)\varphi}\,\beta^n\leqslant\frac{2n}{p^2}\int_{B(r)}|f|^p\,dd^c\Big[\Big(1-\frac{\varphi}{r}\Big)^2e^{(1-C)\varphi}\Big]\wedge\beta^{n-1}.$$

An easy calculation gives, on the other hand, the following bound on $B(r)=\{\varphi<r\}$, uniform in r:

$$dd^c\Big[\Big(1-\frac{\varphi}{r}\Big)^2e^{(1-C)\varphi}\Big]\leqslant C_3\Big(1+\frac{1}{r}\Big)^2e^{-C\varphi}\beta,\qquad r>0.$$

Passing to the limit as $r\to+\infty$, we therefore obtain

$$\int_X|f|^{p-2}|df|_\beta^2\,e^{(1-C)\varphi}\,\beta^n\leqslant C_3\int_X|f|^pe^{-C\varphi}\beta^n.$$

Property 11.3 (b) now follows by applying Hölder's inequality to the measure $|f|^pe^{-C\varphi}\beta^n$ and the functions

$$|f|^{-p}|df|_\beta^p\exp(\frac{p}{2}\varphi)\quad\text{and}\quad 1,$$

with conjugate exponents $(\frac{2}{p},\frac{2}{2-p})$. $\square$

The existence of nonconstant holomorphic functions in $L^p_\varphi(X)$ will follow from the classical estimates of L. Hörmander [Hö1] for the operator $\overline{\partial}$.

Proposition 11.5. – *The following properties hold.*

(a) *Let $\tau\in L^1_{\mathrm{loc}}(X)$ be a function such that*

$$i\,\partial\overline{\partial}\tau+\mathrm{Ricci}(\beta)\geqslant\lambda\beta$$

where λ is a continuous function >0 on X. Let u be a $(0,1)$-form with L^2_{loc} coefficients on X such that $\overline{\partial}u=0$ and

$$\int_X\lambda^{-1}|u|^2e^{-\tau}\,\beta^n<+\infty.$$

Then there exists a function $g\in L^2_{\mathrm{loc}}(X)$ such that $\overline{\partial}g=u$ and

$$\int_X|g|^2e^{-\tau}\,\beta^n\leqslant\int_X\lambda^{-1}|u|^2e^{-\tau}\,\beta^n<+\infty.$$

(b) *Let* ψ, c, A *be the data of* 9.1'(b'). *If* ρ *is psh on* X *and* u *satisfies* $\overline{\partial} u = 0$ *and*

$$\int_X |u|^2\, e^{-\rho-\psi}\, \beta^n < +\infty,$$

then there exists $g \in L^2_{\mathrm{loc}}(X)$ *such that* $\overline{\partial} g = u$ *and*

$$\int_X |g|^2 e^{-\rho-\psi-2\varphi}\, \beta^n \leqslant 4 \int_X |u|^2 e^{-\rho-\psi}\, \beta^n.$$

(c) *Let* $\{x_1, x_2, \ldots, x_m\} \subset X$ *be a finite set, and let* ρ *be a psh function on* X *such that* $\mathrm{e}^{-\rho}$ *is integrable in a neighborhood of* $x_1, x_2, \ldots, x_m$. *Then there exists a holomorphic function* f *having a prescribed jet of order* s *at each point* x_j *and such that*

$$\int_X |f|^2 e^{-\rho-\psi-C_1\varphi}\, \beta^n < +\infty, \qquad \text{where } C_1 \geqslant 0.$$

In particular, if $\rho \equiv 0$, *one obtains* $f \in A^b_\varphi(X)$ *with* $b = \frac{2c}{1+c}$ *and* c *as in* 9.1'(b').

(d) $A^b_\varphi(X)$ *is dense in* $\mathcal{O}(X)$ *for the topology of uniform convergence on compact subsets.*

Proof.

(a) is classical; see, for example, H. Skoda [Sk3].

(b) Apply (a) with $\tau = \rho + \psi + 2\log(1 + e^\varphi)$. Since $\varphi \geqslant 0$, one has $\tau \leqslant \rho + \psi + 2\varphi + \log 4$, and assumption 9.1'(b') yields

$$i\,\partial\overline{\partial}\tau + \mathrm{Ricci}(\beta) \geqslant \frac{e^\varphi\, dd^c\varphi}{1 + e^\varphi} + \frac{e^\varphi\, d\varphi \wedge d^c\varphi}{(1 + e^\varphi)^2} \geqslant \lambda\beta$$

with $\lambda = (1 + e^\varphi)^{-2}$. Estimate (b) follows.

(c) follows from (b) by a classical argument due to Bombieri and Skoda [Sk1]. Let $U_1, \ldots, U_m$ be pairwise disjoint open neighborhoods of $x_1, \ldots, x_m$ on which $e^{-\rho}$ is locally integrable. Equip U_j with a system of local coordinates $z^{(j)} = (z^j_1, z^j_2, \ldots, z^j_n)$ centered at x_j, and set

$$\rho_1 = \rho + (n+s)\left[\sum_{j=1}^m \chi_j \log|z^{(j)}|^2 + C_j\varphi\right]$$

where χ_j is a function $\geqslant 0$ of class $\mathcal{C}^\infty$, compactly supported in U_j and equal to 1 in a neighborhood of x_j, and C_j is a sufficiently large constant $\geqslant 0$ for ρ_1

to be psh on X. The constant $n+s$ is chosen here so that the jet of order s of a function g of class $\mathcal{C}^\infty$ that is locally integrable with respect to the measure $e^{-\rho_1}\,\beta^n$ necessarily vanishes at the points x_j. Now let $P_j(z^{(j)})$ be a polynomial of degree $\leqslant s$ having the prescribed jet at x_j. Set

$$h = \sum_{j=1}^{m} \chi_j\, P_j(z^{(j)}).$$

The $(0,1)$-form

$$u := \overline{\partial} h = \sum_{j=1}^{m} \overline{\partial}\chi_j\ P_j(z^{(j)}).$$

is of class $\mathcal{C}^\infty$, vanishes in a neighborhood of $x_1, \ldots, x_m$, and by construction

$$\int_X |u|^2 e^{-\rho_1-\psi}\,\beta^n < +\infty;$$

here we used the fact that ψ is locally bounded. By (b), there exists $g \in \mathcal{C}^\infty(X)$ such that $\overline{\partial} g = u$ and

$$\int_X |g|^2 e^{-\rho_1-\psi-2\varphi}\,\beta^n < +\infty.$$

The function $f = h - g$ then has the required properties. If $\rho \equiv 0$, we may write

$$|f| = \Big[|f| \exp\Big(-\frac{1}{2}\psi\Big)\Big]\ \exp\Big(\frac{1}{2}\psi\Big)$$

where $|f|\exp(-\frac{1}{2}\psi) \in L^2_\varphi(X)$ and $\exp(\frac{1}{2}\psi) \in L^{2c}_\varphi(X)$. It follows from (11.4) that $f \in L^b_\varphi(X)$.

(d) follows from (b) exactly as in Lemma 4.3.1 of [Hö2]. □

We now use Proposition 11.5 to construct many holomorphic functions on X, and thereby obtain a partial embedding of X into $\mathbb{C}^N$. Let $x_0 \in X$ be a fixed point. By 11.5 (c), applied with $\rho \equiv 0$, there exist functions $f_1, \ldots, f_n \in A^b_\varphi(X)$ such that

$$df_1 \wedge \ldots \wedge df_n(x_0) \neq 0.$$

In particular, $f_1, \ldots, f_n$ are algebraically independent in $A_\varphi(X)$. Theorem 8.5 (b) therefore applies, giving:

Proposition 11.6. – *The field $K_\varphi(X)$ of φ-rational functions is a finitely generated extension of $\mathbb{C}$, of transcendence degree n.* □

As we shall see, it is easy to deduce from the preceding results the existence of a morphism $F : X \to M$ from X to an algebraic variety M of dimension n which, outside an algebraic hypersurface $S \subset X$, is an isomorphism from $X \smallsetminus S$ onto an open subset of M. The main remaining difficulty will be to prove that F is quasi-surjective, i.e. that $F(X \smallsetminus S)$ is a Zariski-open subset of M.

Proposition 11.7. – *The following existence properties hold.*

(a) *There exists a function $f_{n+1} \in A^0_\varphi(X)$ such that*

$$f_{n+1}(x_0) = 1 \quad \textit{and} \quad \int_X |f_{n+1}|^2 \, |df_1 \wedge \ldots \wedge df_n|_\beta^{-2} \, \exp(-2\psi - C\varphi) \, \beta^n < +\infty.$$

In particular, $\{x \in X \,;\, df_1 \wedge \ldots \wedge df_n(x) = 0\} \subset f_{n+1}^{-1}(0)$.

(b) *There exist functions $f_{n+2}, \ldots, f_N \in A^0_\varphi(X)$ and an irreducible algebraic subvariety $M \subset \mathbb{C}^N$ of dimension n such that the morphism $F = (f_1, \ldots, f_N)$ maps X into M and is an analytic isomorphism from $X \smallsetminus f_{n+1}^{-1}(0)$ onto a smooth open subset of M.*

Proof.

(a) It suffices to apply 11.5 (c) to the psh function

$$\rho = \psi + \log |df_1 \wedge \ldots \wedge df_n|^2.$$

If Z is the zero divisor of the holomorphic n-form $df_1 \wedge \ldots \wedge df_n$, viewed as a section of $\bigwedge^n T^*X$, then assumption 9.1'(b') indeed gives:

$$dd^c\rho = dd^c\psi + 2\,\mathrm{Ricci}(\beta) + 4\pi\,[Z] \geqslant 0,$$

and ρ is continuous in a neighborhood of x_0. Hence there exists $f_{n+1} \in \mathcal{O}(X)$ such that $f_{n+1}(x_0) = 1$ and satisfying the stated L^2 estimate. This estimate implies that f_{n+1} vanishes on the support of Z, and by Lemma 11.3 (b) one has $|df_j| \in L^b_\varphi(X)$, whence

$$|f_{n+1}| \leqslant \Big(|f_{n+1}|\,|df_1 \wedge \ldots \wedge df_n|^{-1}\,e^{-\psi}\Big)\,|df_1| \cdots |df_n|\,e^{\psi} \in L^0_\varphi(X).$$

(b) We first construct, by induction on j, functions $f_1, \ldots, f_{N_j} \in A^0_\varphi(X)$ with $N_0 = n + 1 < N_1 < N_2 < \cdots$ (this has been done for $j = 0$).

By 11.6, the image of the morphism

$$F_j = (f_1, \ldots, f_{N_j}) : X \to \mathbb{C}^{N_j}$$

is contained in an irreducible algebraic variety $M_j \subset \mathbb{C}^{N_j}$ of dimension n. Let $\tilde{M}_j$ be the normalization of M_j. There is a commutative diagram

$$\begin{array}{ccccc}
\tilde{M}_j & \hookrightarrow & \mathbb{C}^{\tilde{N}_j} \ni & (z_1, \ldots, z_{\tilde{N}_j}) \\
\downarrow & & \downarrow & \downarrow \\
M_j & \hookrightarrow & \mathbb{C}^{N_j} \ni & (z_1, \ldots, z_{N_j})\,.
\end{array}$$

Since the variety X is smooth (and hence normal), the morphism $F_j : X \to M_j$ lifts to a morphism

$$\tilde{F}_j = (f_1, \ldots, f_{\tilde{N}_j}) : X \to \tilde{M}_j.$$

By construction, the coordinate functions $z_{N_j+1}, \ldots, z_{\tilde{N}_j}$ (resp. $f_{N_j+1}, \ldots, f_{\tilde{N}_j}$) are integral over the ring $\mathbb{C}[z_1, \ldots, z_{N_j}]/I(M_j)$ (resp. over $\mathbb{C}[f_1, \ldots, f_{N_j}]$), hence $f_{N_j+1}, \ldots, f_{\tilde{N}_j} \in A^0_\varphi(X)$ by 11.2 (d). Moreover, the restriction

$$\tilde{F}_j : X \smallsetminus f_{n+1}^{-1}(0) \to \tilde{M}_j$$

is étale, since $df_1 \wedge \ldots \wedge df_n \neq 0$ on $X \smallsetminus f_{n+1}^{-1}(0)$ by (a). Since $\tilde{M}_j$ is locally irreducible, the image $\tilde{F}_j(X \smallsetminus f_{n+1}^{-1}(0))$ is necessarily contained in the set of smooth points of $\tilde{M}_j$. If $\tilde{F}_j$ is injective on $X \smallsetminus f_{n+1}^{-1}(0)$, the construction is complete with $F = \tilde{F}_j$, $M = \tilde{M}_j$, $N = \tilde{N}_j$.

Otherwise, let $z_1 \neq z_2$ be two points in $X \smallsetminus f_{n+1}^{-1}(0)$ such that $\tilde{F}_j(z_1) = \tilde{F}_j(z_2)$. Proposition 11.5 (c) shows that there exists a function $g \in A^b_\varphi(X)$ such that $g(z_1) \neq g(z_2)$. Set $N_{j+1} = \tilde{N}_j + 1$, $f_{\tilde{N}_j+1} = g$. By 11.6, g is algebraic over $\mathbb{C}[f_1, \ldots, f_{\tilde{N}_j}]$, and therefore satisfies an irreducible equation of the form

$$\text{(11.8)} \qquad \sum_{k=0}^{d} a_k(\tilde{F}_j)\, g^k = 0, \qquad a_k \in \mathbb{C}[f_1, \ldots, f_{\tilde{N}_j}], \quad a_d(\tilde{F}_j) \not\equiv 0.$$

Since $\tilde{F}_j$ is étale in neighborhoods of z_1 and z_2, there exist points w_1 near z_1 and w_2 near z_2 such that $\tilde{F}_j(w_1) = \tilde{F}_j(w_2) \notin a_d^{-1}(0)$ and $g(w_1) \neq g(w_2)$. This implies that equation (11.8) has degree $d \geqslant 2$. Since $K_\varphi(X)$ is a finite-degree extension of $\mathbb{C}(f_1, \ldots, f_n)$, the process must terminate after finitely many steps. □

12 Quasi-surjectivity of the embedding.

We retain the notation of Proposition 11.7. The purpose of this section is to prove that the image of the morphism $F : X \smallsetminus f_{n+1}^{-1}(0) \to M$ is a Zariski-open subset of M. Let $Q \in \mathbb{C}[z_1, \ldots, z_N]$ be a polynomial that is not identically zero on M, divisible by z_{n+1}, and such that the hypersurface $Q^{-1}(0)$ contains the singular locus M_{sing}. Set

$$\check{M} = M \smallsetminus Q^{-1}(0), \qquad \check{X} = X \smallsetminus Q(F)^{-1}(0) \subset X \smallsetminus f_{n+1}^{-1}(0),$$

so that $\check{M}$ is smooth and the restricted morphism

$$\check{F} : \check{X} \to \check{M}$$

is an isomorphism from $\check{X}$ onto the open set $\Omega = \check{F}(\check{X})$. The variety $\check{M}$ may be (and will be) identified with an affine algebraic subvariety of $\mathbb{C}^{N+1}$ via the map $\check{M} \to \mathbb{C}^{N+1}$ defined by

$$(z_1, \ldots, z_N) \mapsto (z_1, \ldots, z_N, z_{N+1} = Q(z_1, \ldots, z_N)^{-1});$$

the morphism $\check{F} : \check{X} \to \check{M} \subset \mathbb{C}^{N+1}$ is then given by $\check{F} = (F, Q(F)^{-1})$. One of the crucial points in the argument is to show that the closed positive current $\check{F}_* dd^c\varphi$ extends from the open set $\Omega = \check{F}(\check{X})$ to the entire variety $\check{M}$.[53] For this we need precise mass estimates, which are provided by the following lemma.

Lemma 12.1. – *Let $G = (g_1, \ldots, g_m) \in \left[A^0_\varphi(X)\right]^m$ and $\gamma = dd^c \log(1 + |G|^2)$. Then for every integer $k \geqslant 0$, one has:*

(a) $\displaystyle\int_X (dd^c\varphi)^{n-k} \wedge \gamma^k < +\infty, \qquad 0 \leqslant k \leqslant n,$

(b) $\displaystyle\int_{B(r)} d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{n-k-1} \wedge \gamma^k \leqslant Cr, \qquad 0 \leqslant k \leqslant n-1,$

where C is a constant $\geqslant 0$.

Proof. Apply Theorem 2.2 (c) with $\rho = \varphi - r$, $\Omega = \{\rho < 0\} = B(r)$ and $V_1 = \ldots = V_k = \log(1 + |G|^2) \geqslant 0$. This gives

$$\int_{B(r)} \beta_k \wedge \gamma^k \leqslant C_3 \int_{B(r)} \big(\log(1 + |G|^2)\big)^k \, \beta_0$$

where, for $a > 0$ and $k \geqslant 0$, we have set:

$$\beta_k = (r - \varphi)^{k+a} \, (dd^c\varphi)^{n-k} + (k + a)(r - \varphi)^{k-1+a} \, d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{n-k-1}.$$

We have:

$$\beta_0 = 2(r - \varphi)^a \, (dd^c\varphi)^n + \frac{1}{2(1 + a)} \, dd^c\beta_1.$$

Stokes' theorem therefore gives, for every $r > 0$:

$$\int_{B(r)} \beta_0 = 2 \int_{B(r)} (r - \varphi)^a \, (dd^c\varphi)^n \leqslant 2 \, r^a \, \mathrm{Vol}(X).$$

On the other hand, the function $t \mapsto (\log(e^k + t))^k$ is concave on $[0, +\infty[$. Hence, for every $p > 0$, we obtain Jensen's inequality:

$$\frac{\int_{B(r)} \big(\log(e^k + |G|^p)\big)^k \, \beta_0}{\int_{B(r)} \beta_0} \leqslant \left\{ \log \left[e^k + \frac{\int_{B(r)} |G|^p \, \beta_0}{\int_{B(r)} \beta_0} \right] \right\}^k.$$

[53] **[P]** The current is extended to $\check{M}$, as in the published statement.

Since $g_j \in A^0_\varphi(X)$ [cf. Definition 11.1], there exist $p > 0$ sufficiently small and $C_4, C_5 \geqslant 0$ sufficiently large such that

$$\int_{B(r)} |G|^p\,\beta_0 \leqslant \exp(C_4 r + C_5).$$

The preceding inequalities then imply

$$\int_{B(r)} \beta_k \wedge \gamma^k \leqslant C_6 \int_{B(r)} \big(\log(e^k + |G|^p)\big)^k\,\beta_0 \leqslant C_7\, r^{k+a}.$$

In view of the definition of β_k, this proves Lemma 12.1 after substituting $2r$ for r. □

Equip $\mathbb{C}^{N+1}$ and $\check{M} \subset \mathbb{C}^{N+1}$ with the Fubini–Study metric $\omega = dd^c \log(1+|z|^2)$. We then have the following extension theorem, whose proof is inspired by H. Skoda [Sk5] and H. El Mir [EM]; see also the survey article by N. Sibony [Sib].

Proposition 12.2. – *Let T be the trivial extension to $\check{M}$ of the current $\check{F}_* dd^c\varphi$, defined by*

$$T = \check{F}_* dd^c\varphi \quad \text{on } \check{F}(\check{X}), \qquad T = 0 \quad \text{on } \check{M} \smallsetminus \check{F}(\check{X}).$$

Then T is a closed positive current on $\check{M}$ with finite total mass $\int_{\check{M}} T \wedge \omega^{n-1}$.

Proof. First compute the mass of T:

$$\int_{\check{M}} T \wedge \omega^{n-1} = \int_{\check{F}(\check{X})} \check{F}_* dd^c\varphi \wedge \omega^{n-1} = \int_{\check{X}} dd^c\varphi \wedge (\check{F}^*\omega)^{n-1}.$$

The $(1,1)$-form $\check{F}^*\omega$ is given here by

$$\begin{aligned}\check{F}^*\omega &= dd^c \log(1+|\check{F}|^2) = dd^c \log(1+|F|^2 + |Q(F)|^{-2})\\ &= dd^c \log(1+|Q(F)|^2 + |F|^2|Q(F)|^2).\end{aligned}$$

The finiteness of the mass now follows from Lemma 12.1 (a). For every real 1-form v of class $\mathcal{C}^\infty$ with compact support in $\check{M}$[54] and all multi-indices $J, K \subset \{1, \ldots, N+1\}$ such that $|J| = |K| = n-2$, we now show that

$$I = \int_{\check{M}} dv \wedge T \wedge dz_J \wedge d\overline{z}_K = 0.$$

[54] **[P]** The test form v is defined on $\check{M}$, where the current T lives.

This will prove that $dT = 0$. Let χ be a function of class $\mathcal{C}^\infty$ on $\mathbb{R}$ such that $0 \leqslant \chi \leqslant 1$, $\chi(t) = 1$ if $t < 0$, $\chi(t) = 0$ if $t > 1$, and $|\chi'| \leqslant 2$. [55] By the definition of T, we obtain

$$\begin{aligned} I &= \int_{\check{X}} \check{F}^*(dv) \wedge dd^c\varphi \wedge d\check{F}_J \wedge \overline{d\check{F}_K} \\ &= \lim_{r\to+\infty} \int_{\check{X}} \chi\Big(\frac{\varphi}{r}\Big)\, d(\check{F}^*v) \wedge dd^c\varphi \wedge d\check{F}_J \wedge \overline{d\check{F}_K}\ . \end{aligned}$$

The form $\chi(\frac{\varphi}{r})\, d(\check{F}^*v)$ is supported in $\check{F}^{-1}(\operatorname{Supp} v) \cap \overline{B(r)} \Subset \check{X}$. Integration by parts therefore gives

$$I = \lim_{r\to+\infty} \pm \int_{\check{X}} \check{F}^*v \wedge \chi'\Big(\frac{\varphi}{r}\Big)\, \frac{d\varphi}{r} \wedge dd^c\varphi \wedge d\check{F}_J \wedge \overline{d\check{F}_K}\ .$$

By the Cauchy–Schwarz inequality, the last integral is bounded by $\frac{2}{r}\sqrt{I_1 I_2(r)}$, where[56]

$$\begin{aligned} I_1 &= \int_{\check{X}} dd^c\varphi \wedge \check{F}^*\big(v \wedge v^c \wedge dz_J \wedge d\overline{z}_J\big), \\ I_2(r) &= \int_{\check{F}^{-1}(\operatorname{Supp} v)\cap B(r)} d\varphi \wedge d^c\varphi \wedge dd^c\varphi \wedge d\check{F}_K \wedge \overline{d\check{F}_K}\ . \end{aligned}$$

Since v has compact support in $\check{M}$, there exist constants $C_1, C_2 \geqslant 0$ such that

$$\begin{aligned} &v \wedge v^c \wedge dz_J \wedge d\overline{z}_J \leqslant C_1 \omega^{n-1}, \\ &d\check{F}_K \wedge \overline{d\check{F}_K} = \check{F}^*(dz_K \wedge d\overline{z}_K) \leqslant C_2(\check{F}^*\omega)^{n-2} \quad \text{on } \check{F}^{-1}(\operatorname{Supp} v). \end{aligned}$$

Lemma 12.1 (a) and (b) then give[57]

$$\begin{aligned} I_1 &\leqslant C_1 \int_{\check{X}} dd^c\varphi \wedge (\check{F}^*\omega)^{n-1} < +\infty, \\ I_2(r) &\leqslant C_2 \int_{B(r)} d\varphi \wedge d^c\varphi \wedge dd^c\varphi \wedge (\check{F}^*\omega)^{n-2} \leqslant C\, C_2\, r, \end{aligned}$$

whence

$$|I| \leqslant \lim_{r\to+\infty} \frac{2}{r}\sqrt{I_1 I_2(r)} = 0. \qquad \square$$

By Theorem 15.3 in the appendix, there exist a psh function V and a $(1,0)$-form u of class $\mathcal{C}^\infty$ on $\check{M}$ having the following properties, for suitable constants $C_1, C_2, C_3 \geqslant 0$.

---

[55] **[E]** The cutoff must satisfy $|\chi'| \leqslant 2$. A one-sided bound is not enough for the decreasing cutoff used in the proof.

[56] **[P/E]** Both integrals below are taken over $\check{X}$, and the quadratic form in the first is $v \wedge v^c$. These choices make the Cauchy–Schwarz estimate well defined.

[57] **[P/E]** The first estimate is over $\check{X}$, and both estimates use the pullback $\check{F}^*\omega$ on that space.

Properties 12.3. – *We have the* (in)*equalities*

(a) $dd^c V \geqslant T$;

(b) $V(z) \leqslant C_1 \log(1+|z|^2)$;

(c) $dd^c V - T = \overline{\partial} u$;

(d) $|u|_\omega \leqslant C_2(1+|z|^2)^{C_3}$.

Now consider the function $\tau = V - \check{F}_*\varphi$ defined on the open set $\Omega = \check{F}(\check{X}) \subset \check{M}$. By 12.3 (a), τ is psh on Ω, and moreover $\tau \leqslant V$. Since $\check{F}_*\varphi$ tends to $+\infty$ near $\partial\Omega$, τ tends to $-\infty$ at every point of $\partial\Omega$. Consequently, τ extends to a psh function on $\check{M}$, still denoted τ, such that $\tau = -\infty$ on $\check{M} \smallsetminus \Omega$.

Corollary 12.4. – $\check{M} \smallsetminus \Omega$ *is a closed pluripolar subset of* $\check{M}$. □

The next step is to show that $\check{M} \smallsetminus \Omega$ is in fact an algebraic hypersurface of $\check{M}$. By 12.3 (c) and the definition of T, we have:

$$2i\,\partial\overline{\partial}(V - \check{F}_*\varphi) = \overline{\partial} u \quad \text{on } \Omega,$$

so the $(1,0)$-form h defined by

$$h = \partial(V - \check{F}_*\varphi) + \frac{u}{2i} = \partial\tau + \frac{u}{2i} \tag{12.5}$$

is holomorphic on Ω; since u is of class $\mathcal{C}^\infty$ on $\check{M}$, this also shows that τ is of class $\mathcal{C}^\infty$ on Ω. We shall now prove that h extends to a rational meromorphic 1-form on $\check{M}$. This will follow essentially from estimates 12.3 (b,d) and the algebraicity theorem 8.5. By construction of F and $\check{X}$, the forms $(df_1,\ldots,df_n)$ define a global frame of $T^*\check{X}$. The forms $(dz_1,\ldots,dz_n)$ therefore also form a frame of $T^*\check{M}$ over the open set $\Omega = \check{F}(\check{X})$, and we may write

$$h = \sum_{j=1}^n h_j dz_j$$

with functions $h_j \in \mathcal{O}(\Omega)$. The idea of the argument is to verify that the functions $h_j \circ \check{F}$ have φ-polynomial growth, using the upper bound on $\tau = V - \check{F}_*\varphi$ supplied by 12.3 (b). The fact that we have no lower bound on τ introduces an additional difficulty, which we shall circumvent by seeking only an estimate for the functions $\exp(\frac{1}{2}\tau \circ \check{F})\ |h_j(\check{F})|$.

Lemma 12.6. – *Consider on* $\check{X}$ *the exhaustion function*[58]

$$\check{\varphi} = \log(1+e^\varphi) + \log(1+|\check{F}|^2)$$

[58] **[P]** Both $\check{\varphi}$ and $\check{\alpha}$ are defined on $\check{X}$. The published formula also supplies the missing term $dd^c \log(1+e^\varphi)$ in $\check{\alpha}$.

and the associated metric

$$\check\alpha = dd^c\check\varphi = dd^c\log(1+e^{\varphi}) + \check F^*\omega.$$

The following properties then hold for constants $p>0$ sufficiently small and C_4, C_5 sufficiently large.

(a) $\displaystyle\int_{\check X} \check\alpha^n < +\infty;$

(b) $\displaystyle\int_{\check X} e^{\tau\circ\check F - C_4\check\varphi}\ \check F^*(ih\wedge\overline{h})\wedge\check\alpha^{n-1} < +\infty;$

(c) $\displaystyle\int_{\check X} \Big[\exp\Big(\frac12\tau\circ\check F\Big)\ |Q(F)|^{C_4+1}\ |h_j(\check F)|\Big]^p\ e^{-C_5\varphi}\ \beta^n < +\infty.$

Proof.

(a) follows immediately from Lemma 12.1 upon observing that

$$dd^c\log(1+e^{\varphi}) = \frac{e^{\varphi}\ dd^c\varphi}{1+e^{\varphi}} + \frac{e^{\varphi}\ d\varphi\wedge d^c\varphi}{(1+e^{\varphi})^2} \leqslant dd^c\varphi + e^{-\varphi}\ d\varphi\wedge d^c\varphi.$$

(b) Estimate 12.3 (b) implies[59]

$$\tau = V - \check F_*\varphi \leqslant V \leqslant C_1\log(1+|z|^2), \tag{12.7}$$

so the function $\theta = \log(1+e^{\tau\circ\check F})$ satisfies the upper bound

$$\theta \leqslant \log\big(1+(1+|\check F|^2)^{C_1}\big) \leqslant C_1\check\varphi + \log 2.$$

Corollary 7.3 applied to $(\check X, \check\varphi)$ then gives

$$\int_{\check X} i\partial\overline\partial\theta\wedge\check\alpha^{n-1} < +\infty.$$

A direct computation also gives[60]

$$i\partial\overline\partial\theta = \check F^*\Big(\frac{i\partial\overline\partial\tau}{1+e^{-\tau}} + \frac{ie^{\tau}\ \partial\tau\wedge\overline\partial\tau}{(1+e^{\tau})^2}\Big) \geqslant \frac12\ \check F^*(ie^{\tau}\ \partial\tau\wedge\overline\partial\tau)\ e^{-2C_1\check\varphi},$$

and it follows that

$$\int_{\check X} e^{\tau\circ\check F - 2C_1\check\varphi}\ \check F^*(i\partial\tau\wedge\overline\partial\tau)\wedge\check\alpha^{n-1} < +\infty.$$

[59] **[P]** In (12.7), φ is pushed forward by $\check F$. A pullback here would have the wrong domain.
[60] **[P]** The published estimate includes the factor i in $\check F^*(i\partial\tau\wedge\overline\partial\tau)$, making the form positive.

On the other hand, by the definition of $\check\alpha$ we have $\check\alpha \geqslant \check F^*\omega$. Estimate 12.3 (d) therefore gives

$$|\check F^* u|_{\check\alpha} \leqslant (|u|_\omega)\circ \check F \leqslant C_2(1+|\check F|^2)^{C_3} \leq C_2\, e^{C_3\check\varphi},$$

$$\int_{\check X} |\check F^* u|^2_{\check\alpha}\, e^{\tau\circ\check F-(C_1+2C_3)\check\varphi}\, \check\alpha^n \leqslant C_2^2\int_{\check X}\check\alpha^n < +\infty.$$

Property (b) now follows from the definition of $h = \partial\tau + \frac{u}{2i}$ and the identity

$$|\check F^* u|^2_{\check\alpha}\, \check\alpha^n = n\, \check F^*(iu\wedge\overline{u})\wedge\check\alpha^{n-1}.$$

(c) The trivial inequality

$$dd^c\log(1+e^\varphi) \geqslant \frac{1}{2}\, dd^c\varphi + \frac{1}{4}\, e^{-\varphi}\, d\varphi\wedge d^c\varphi$$

successively implies

$$\begin{aligned}\check\alpha^{n-1} &\geqslant \frac{1}{2^{n-1}}\,(dd^c\varphi)^{n-1} + \frac{(n-1)\, e^{-\varphi}}{2^n}\,(dd^c\varphi)^{n-2}\wedge d\varphi\wedge d^c\varphi\\ &\geqslant 2^{-n}e^{-n\varphi}\,(dd^c e^\varphi)^{n-1} = 2^{-n}e^{-n\varphi}\,\beta^{n-1},\end{aligned}$$

$$\check F^*(ih\wedge\overline{h})\wedge\check\alpha^{n-1} \geqslant 2^{-n}e^{-n\varphi}\,|\check F^* h|^2_\beta\,\beta^n.$$

On the other hand, by the definition of $\check\varphi$, we have

$$\begin{aligned}\check\varphi &= \log(1+e^\varphi) - 2\log|Q(F)| + \log\big(1+|Q(F)|^2+|Q(F)|^2|F|^2\big)\\ &\leqslant \varphi - 2\log|Q(F)| + C_6\log(1+|F|^2) + C_7,\end{aligned}$$

where $C_6 = 1+\deg(Q)$.[61] Inequality (b) therefore gives

$$\int_{\check X} e^{\tau\circ\check F - C_4\varphi}\,|Q(F)|^{2C_4}\,(1+|F|^2)^{-C_4C_6}e^{-n\varphi}\,|\check F^* h|^2_\beta\,\beta^n < +\infty$$

and since $|F|\in L^0_\varphi(X)$, it follows that

$$\exp\Big(\frac{1}{2}\tau\circ\check F\Big)\,|Q(F)|^{C_4}\,|\check F^* h|_\beta \in L^0_\varphi(X).$$

Moreover, the function h_j can be written as

$$h_j = (-1)^{j-1}\,\frac{h\wedge dz_1\wedge\ldots\wedge\widehat{dz_j}\wedge\ldots\wedge dz_n}{dz_1\wedge\ldots\wedge dz_n}.$$

[61] **[E]** The constant here depends on $\deg Q$. Both source versions mistakenly refer to a different polynomial.

Thus

$$|h_j(\check{F})| \leqslant |\check{F}^*h|_\beta \, |df_1|_\beta \ldots \widehat{|df_j|_\beta} \ldots |df_n|_\beta \, |df_1 \wedge \ldots \wedge df_n|_\beta^{-1},$$

and since[62]

$$|df_1|_\beta \, , \; \ldots \, , \; |df_n|_\beta \in L^0_\varphi(X) \qquad \text{[Lemma 11.3 (b)], and}$$

$$|f_{n+1}| \, |df_1 \wedge \ldots \wedge df_n|_\beta^{-1} \in L^0_\varphi(X) \qquad \text{[inequality 11.7 (a)],}$$

we obtain

$$\exp\Big(\frac{1}{2}\tau \circ \check{F}\Big) \, |Q(F)|^{C_4} \, |f_{n+1}| \, |h_j \circ \check{F}| \in L^0_\varphi(X).$$

By assumption Q is divisible by z_{n+1}, i.e. $Q = z_{n+1}R$. Property (c) is then obtained by multiplying the function above by $|R(F)| \in L^0_\varphi(X)$. $\square$

In order to work on X rather than on $\check{X}$, we shall need the following elementary extension lemma.

Lemma 12.8. – *Let $S = g^{-1}(0)$ be a hypersurface of X, and let θ be a psh function on $X \smallsetminus S$ such that $e^\theta \in L^1_{\mathrm{loc}}(X)$. Then $\theta + \log|g|^2$ extends to a psh function on X.*

Proof. It suffices to show that $\theta + \log|g|^2$ is bounded above near every regular point of S. We may therefore assume that X is an open subset of $\mathbb{C}^n$ containing the closed unit polydisc $\overline{\Delta}^n$, and that $S = \{z_1 = 0\}$. The mean-value inequality applied to the polydisc

$$(z_1 + |z_1| \, \Delta) \times \Delta^{n-1} \subset X \smallsetminus S$$

at every point $z \in \Delta^n$, $0 < |z_1| < \frac{1}{2}$, implies

$$e^{\theta(z)} \leqslant \frac{1}{\pi^n |z_1|^2} \int_{\Delta^n} e^\theta \, d\lambda.$$

It follows that the function $\theta + \log|z_1|^2$ is bounded above near S. $\square$

Proposition 12.9. – *The 1-form $h = \sum_{1 \leqslant j \leqslant n} h_j dz_j$ extends to a rational meromorphic 1-form on $\check{M}$.*[63]

Proof. Since $\check{X} = X \smallsetminus Q(F)^{-1}(0)$, Lemmas 12.6 (c) and 12.8 show that

$$p \, \log\Big[\exp\Big(\frac{1}{2}\tau \circ \check{F}\Big) \, |Q(F)|^{C_4+1} \, |h_j \circ \check{F}| \,\Big] + \log|Q(F)|^2$$

[62] **[P]** The sequence is $df_1, \ldots, df_n$. The final differential is accidentally repeated in the web TeX.

[63] **[P]** The rational meromorphic form extends to $\check{M}$, the ambient variety in this section.

extends to a psh function on X. Thus there exist an integer $s > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small such that, if g denotes the holomorphic function on X defined by[64]

$$g = Q(F)^s h_j(\check{F}),$$

then the function $\frac{1}{2}\tau \circ \check{F} + \log|g|$ is psh on X, and

$$\int_X \exp\Big[\varepsilon\Big(\frac{1}{2}\tau \circ \check{F} + \log|g|\Big) - C_8\varphi\Big]\,\beta^n < +\infty.$$

Arguing as in Lemma 11.3 (a), we consequently obtain

$$\limsup_{r\to+\infty} \frac{1}{r}\,\mu_r\Big[\Big(\frac{1}{2}\tau \circ \check{F} + \log|g|\Big)_+\Big] \leqslant \frac{C_8 - n}{\varepsilon}\,\mathrm{Vol}(X) < +\infty.$$

Let $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ be a polynomial such that $\deg_{X_\ell} P \leqslant k_\ell$, and let θ be the function defined by

$$\theta = \log|P(g, f_1, \ldots, f_n)| + k_0\Big(\frac{1}{2}\tau \circ \check{F} + C_1 \log|Q(F)|\Big).$$

By estimate (12.7) and the preceding results, θ is psh on X and satisfies an estimate[65]

$$\theta_+ \leqslant \sum_{j=1}^n k_j \log_+|f_j| + k_0\left[\Big(\frac{1}{2}\tau \circ \check{F} + \log|g|\Big)_+ + C_9 \log_+|F|\right] + C_{10}.$$

By Corollary 7.3, we obtain the upper bound

$$\int_X dd^c\theta \wedge \alpha^{n-1} \leqslant C_{11}k_0 + \sum_{j=1}^n k_j \delta_\varphi(f_j)$$

with a constant $C_{11} \geqslant 0$. If $a \in X$, it follows that

$$\mathrm{ord}_a\, P(g, f_1, \ldots, f_n) \leqslant C_{12}(k_0 + k_1 + \cdots + k_n), \qquad C_{12} \geqslant 0.$$

The argument of Theorem 8.5 then shows that g is algebraic over $\mathbb{C}(f_1, \ldots, f_n)$, and hence so is the function $h_j(\check{F}) = Q(F)^{-s}g$. Consequently h_j is algebraic over $\mathbb{C}(z_1, \ldots, z_n)$, i.e. h_j satisfies an equation

$$\sum_{\ell=0}^d a_\ell(z_1, \ldots, z_n)\, h_j^\ell = 0, \qquad a_\ell \in \mathbb{C}[z_1, \ldots, z_n], \quad a_d \not\equiv 0.$$

[64] **[P/E]** The functions in this argument are composed with $\check{F}$, and each estimate is written on the space where its terms are defined.

[65] **[P/E]** The bound includes the complete positive-part term and $C_9 \log_+|F|$. In the following sum, the index runs from 1 to n.

The element $a_d h_j$ is therefore integral over $\mathbb{C}[z_1, \ldots, z_n]$; we deduce an upper bound

$$|a_d(z)h_j(z)| \leqslant C_{13}(1+|z|)^{C_{14}}.$$

Since h_j is holomorphic on the open set $\Omega \subset \check{M}$ and the complement $\check{M} \smallsetminus \Omega$ is pluripolar, $a_d h_j$ extends to a polynomial on $\check{M}$. Consequently $h = \sum h_j dz_j$ extends to a rational meromorphic 1-form on $\check{M}$. □

Proposition 12.10. – *Let Ω_1 be the largest Zariski-open subset of $\check{M}$ on which h is holomorphic. Then $\Omega = \Omega_1$.*

Proof. Clearly $\Omega \subset \Omega_1$. To obtain the reverse inclusion, first let us show that τ is of class $\mathcal{C}^\infty$ on Ω_1. We know that equation (12.5)

$$\partial\tau = h - \frac{u}{2i}$$

holds on Ω, and that $v := h - \frac{u}{2i} \in \mathcal{C}^\infty_{1,0}(\Omega_1)$, since $u \in \mathcal{C}^\infty_{1,0}(\check{M})$. Thus $v + \overline{v} = d\tau$ on Ω, whence $d(v+\overline{v}) = 0$ on Ω_1 by continuity. Let $(\Omega_{1,j})_{j\in J}$ be a covering of Ω_1 by simply connected open sets. There exist functions $\tau_j \in \mathcal{C}^\infty(\Omega_{1,j})$ such that $d\tau_j = v + \overline{v}$ on $\Omega_{1,j}$. The function $\tau - \tau_j$ is then locally constant on $\Omega_{1,j} \cap \Omega$, hence constant, because $\Omega_{1,j} \cap \Omega = \Omega_{1,j} \smallsetminus (\check{M} \smallsetminus \Omega)$ is connected. It follows that $\tau \in \mathcal{C}^\infty(\Omega_1)$, and since $\tau = -\infty$ on $\check{M} \smallsetminus \Omega$, we obtain $\Omega_1 \smallsetminus \Omega = \emptyset$. □

Corollary 12.11. – *$F(X \smallsetminus f_{n+1}^{-1}(0))$ is a Zariski-open subset of M.*

Proof. If x is any point of the open set $X \smallsetminus f_{n+1}^{-1}(0)$, then $F(x) \notin M_{\mathrm{sing}} \cup \{z_{n+1} = 0\}$, so there exists a polynomial Q_x divisible by z_{n+1} and vanishing on M_{sing}, such that $Q_x(F(x)) \neq 0$. By Proposition 12.10,[66] $\Omega_x := F(X \smallsetminus Q_x(F)^{-1}(0))$ is a Zariski-open subset of M, and hence so is the union

$$\bigcup_{x \in X \smallsetminus f_{n+1}^{-1}(0)} \Omega_x = F(X \smallsetminus f_{n+1}^{-1}(0)). \qquad \square$$

Since $X \smallsetminus f_{n+1}^{-1}(0)$ is Stein, as is its biholomorphic image under F, we see in fact that the complement $M \smallsetminus F(X \smallsetminus f_{n+1}^{-1}(0))$ is necessarily an algebraic hypersurface of M.

13 Proof of the algebraicity criterion (smooth case).

By Proposition 11.7 (a), to every point $x_0 \in X$ we can associate a morphism

$$F^{(0)} = (f_1^{(0)}, \ldots, f_{N_0}^{(0)}) \in \left[A^0_\varphi(X)\right]^{N_0}$$

[66] **[E]** The result invoked here is Proposition 12.10, not a corollary.

and a function $g_0 = f_{n+1}^{(0)}$ such that the open set $X \smallsetminus g_0^{-1}(0) \ni x_0$ is biholomorphic under $F^{(0)}$ to a Zariski-open subset of an algebraic variety in $\mathbb{C}^{N_0}$. There is therefore a countable covering of X by such open sets $X \smallsetminus g_k^{-1}(0)$, associated with morphisms $F^{(k)} : X \to \mathbb{C}^{N_k}$. Consider the product morphism

$$F_k = F^{(0)} \times F^{(1)} \times \cdots \times F^{(k)} : X \to \mathbb{C}^{N_0+\cdots+N_k}.$$

By Theorem 8.5, the image $F_k(X)$ is contained in an irreducible algebraic variety $M_k \subset \mathbb{C}^{N_0\,+\,\cdots\,+\,N_k}$ of dimension n, and Corollary 12.11 shows that $F_k(X \smallsetminus g_j^{-1}(0))$ is a Zariski-open subset of M_k if $j \leqslant k$. Set

$$Y_k = \bigcap_{j\leqslant k} g_j^{-1}(0), \qquad X_k = X \smallsetminus Y_k = \bigcup_{j\leqslant k} (X \smallsetminus g_j^{-1}(0)).$$

By construction $F_k : X_k \to F_k(X_k) \subset M_k$ is an isomorphism, and $F_k(X_k)$ is a Zariski-open subset of M_k. We may therefore state:

Proposition 13.1. – *If X satisfies assumptions* 9.1'(a',b')*, then X is the union of an increasing sequence of quasi-affine varieties X_k, where each X_k identifies with a Zariski-open subset of X_{k+1} with the induced algebraic structure.* □

In other words, X does have the structure of a ringed space which is "locally" that of an algebraic variety, but the "Zariski topology" need not be quasi-compact.

Such varieties do in fact exist. It suffices to take for X the (smooth) surface with equation $\sin x = yz$ in $\mathbb{C}^3$, and for Y_k the countable union of lines[67]

$$(\{j\pi\} \times \{0\} \times \mathbb{C}),$$

$j \in \mathbb{Z}$, $|j| > k$. The open set $X_k = X \smallsetminus Y_k$ then identifies with the algebraic variety

$$V_k = \Big\{(x, y, z) \in \mathbb{C}^3\,;\; x\Big(1 - \frac{x^2}{\pi^2}\Big)\cdots\Big(1 - \frac{x^2}{k^2\pi^2}\Big) = yz\Big\}$$

via the map $V_k \hookrightarrow X$ defined by [68]

$$(x, y, z) \mapsto (x, y, z') \quad \text{where} \quad z' = z \prod_{j=k+1}^{+\infty} \Big(1 - \frac{x^2}{j^2\pi^2}\Big),$$

with inclusions $V_k \hookrightarrow V_{k+1}$ given by the algebraic morphisms[69]

$$(x, y, z) \mapsto (x, y, z') \quad \text{where} \quad z' = z\Big(1 - \frac{x^2}{(k+1)^2\pi^2}\Big). \qquad \square$$

[67] **[P]** The published example removes only the single family of lines displayed below. The web TeX adds a second family that is not part of the example.
[68] **[E]** The Euler-product tail is taken once, over $j \geqslant k+1$. Both source versions accidentally repeat the product.
[69] **[W]** The two explicit maps in this paragraph were added in the later web TeX and are useful, so they have been kept.

We shall now show that the sequence (X_k) is necessarily stationary if the cohomology spaces $H^{2q}(X\,;\mathbb{R})$ are finite-dimensional [assumption 9.1'(c')].

Lemma 13.2. – *Let X be a complex manifold* [70] *of dimension n, let Y be an analytic set of dimension $\leqslant p$ in X, and let $d = n - p = \operatorname{codim}_{\mathbb{C}} Y$. Then the relative cohomology space $H^q(X, X \smallsetminus Y\,;\mathbb{R})$ vanishes if $q < 2d$, and*

$$H^{2d}(X, X \smallsetminus Y\,;\mathbb{R}) \simeq \mathbb{R}^J,$$

where $(Y_j)_{j\in J}$ is the family of irreducible components of dimension p of Y.

Proof. We refer, for example, to E. Spanier [Sp] for the elementary arguments from algebraic topology that will be used. We argue by induction on p, the result being trivial for $p = 0$. If $p \geqslant 1$, let Z be the union of the singular locus $Y_{\rm sing}$ and the irreducible components of Y of dimension $< p$, so that $\dim Z \leqslant p - 1$. The exact sequence of the triple is

$$H^q(X, X \smallsetminus Z) \to H^q(X, X \smallsetminus Y) \to H^q(X \smallsetminus Z, X \smallsetminus Y) \to H^{q+1}(X, X \smallsetminus Z).$$

By the induction hypothesis $H^q(X, X \smallsetminus Z) = H^{q+1}(X, X \smallsetminus Z) = 0$ for $q \leqslant 2d$, hence $H^q(X, X\smallsetminus Y) \simeq H^q(X\smallsetminus Z, X\smallsetminus Y)$. After replacing (X, Y) by $(X\smallsetminus Z, Y\smallsetminus Z)$, we may assume that Y is smooth of dimension p.

Y then has a tubular neighborhood U homeomorphic to the normal bundle NY. By the excision theorem, we therefore obtain

$$H^q(X, X \smallsetminus Y) \simeq H^q(U, U \smallsetminus Y) \simeq H^q(NY, N^\bullet Y)$$

where $N^\bullet Y$ is the complement of the zero section of NY. Since the bundle NY has real rank $2d$, the Thom–Gysin isomorphism theorem implies

$$H^q(NY, N^\bullet Y) \simeq H^{q-2d}(Y),$$

and for $q = 2d$, $H^0(Y) \simeq \mathbb{R}^J$. □

We now return to the setting of Proposition 13.1, where $X_k = X \smallsetminus Y_k$, and set $\dim Y_k = p_k$, $d_k = n - p_k$. The exact sequence of the pair $(X, X \smallsetminus Y_k)$ gives

$$H^{2d_k-1}(X \smallsetminus Y_k) \to H^{2d_k}(X, X \smallsetminus Y_k) \to H^{2d_k}(X).$$

Since $X \smallsetminus Y_k$ is isomorphic to an algebraic variety, $H^{2d_k-1}(X \smallsetminus Y_k)$ is finite-dimensional, and the same holds for $H^{2d_k}(X)$ by assumption. Lemma 13.2 therefore shows that Y_k has only finitely many irreducible components of maximal

[70] **[E]** The lemma must be stated for a complex manifold. For a singular analytic variety, its conclusion can fail at a singular point.

dimension p_k. Since Y_k is a decreasing sequence with empty intersection, we see that there exists $\ell > k$ such that $\dim Y_\ell < p_k$. After finitely many steps we shall therefore have $Y_\ell = \emptyset$, hence $X = X_\ell$. Set

$$F = F_\ell = F^{(0)} \times \cdots \times F^{(\ell)}, \qquad M = M_\ell, \qquad N = N_0 + \cdots + N_\ell.$$

The morphism $F : X \to M \subset \mathbb{C}^N$ is then an analytic isomorphism from X onto a Zariski-open subset $\Omega \subset M$. Replacing M by its normalization as in the proof of 11.7 (b), we may assume that M is normal. Since Ω is Stein, the complement $H = M \smallsetminus \Omega$ is necessarily a hypersurface of M. Denote by $K(\Omega) \simeq K(M)$ the field of rational functions on Ω, and by $R(\Omega)$ the ring of algebraic regular functions on Ω. Write

$$F = (f_1, \ldots, f_N) \in \left[A^0_\varphi(X)\right]^N.$$

The comorphism F^* maps $K(\Omega)$ into the field $\mathbb{C}(f_1, \ldots, f_N) \subset K_\varphi(X)$. The following proposition shows that the algebraic structures $(X, K_\varphi(X) \cap \mathcal{O}(X))$ and $(\Omega, R(\Omega))$ are isomorphic.

Proposition 13.3. – *We have the following properties:*

(a) $F^* : K(\Omega) \to K_\varphi(X)$ *is an isomorphism.*

(b) $F^* R(\Omega) = K_\varphi(X) \cap \mathcal{O}(X)$.

Proof.

(a) It suffices to prove the surjectivity of F^*. Now, if $g \in K_\varphi(X)$, then g is algebraic over $\mathbb{C}(f_1, \ldots, f_N)$ by 11.6. Hence $g \circ F^{-1}$ is meromorphic on Ω and algebraic over $K(\Omega)$. It follows that $g \circ F^{-1} \in K(\Omega) = K(M)$, by arguing, for example, as at the end of the proof of 12.9.

(b) follows immediately from (a), provided we verify the equality $R(\Omega) = K(\Omega) \cap \mathcal{O}_{\mathrm{anal}}(\Omega)$. The inclusion $R(\Omega) \subset \ldots$ is clear. Conversely, given $g \in K(\Omega)$ and $x \in \Omega$, write $g = u/v$, where $u, v \in \mathcal{O}_{x,\mathrm{alg}}$, as an irreducible expression for g at x (which is smooth by assumption). This expression is also irreducible in $\mathcal{O}_{x,\mathrm{anal}}$. Since $g \in \mathcal{O}_{x,\mathrm{anal}}$, we therefore have $v(x) \neq 0$, hence $g \in \mathcal{O}_{x,\mathrm{alg}}$ and $g \in R(\Omega)$. □

To complete the proof of Theorem 9.1', it now remains to show that Ω is algebraically isomorphic to an affine algebraic variety; in other words, we must prove the existence of a proper algebraic embedding $\Omega = M \smallsetminus H \to \mathbb{C}^{N'}$. This is easy if M is smooth, but when M is singular the algebraic hypersurface H need not be locally a complete intersection, and in this situation Goodman [Go] gave examples for which the algebra $R(M \smallsetminus H)$ is not finitely generated. We thank N. Mok for pointing out this difficulty to us, which invalidated our original proof.

The argument in [Mok2] consists in observing that Ω is *rationally convex* in the following sense: for every compact set $K \subset \Omega$, the hull

$$\hat{K} = \Big\{x \in \Omega\,;\ |g(x)| \leqslant \sup_K |g| \text{ for every } g \in R(\Omega)\Big\}$$

is compact. Indeed, this follows from 11.5 (d) and the fact that $\Omega \simeq X$ is Stein. We then apply part (b) of the theorem below.

Theorem 13.4. – *Let $M \subset \mathbb{C}^N$ be an affine algebraic variety (possibly singular) of pure dimension n, and let H be an algebraic hypersurface of M. Then $M \smallsetminus H$ is isomorphic to an affine algebraic variety under either of the following two assumptions:*

(a) *H is, (as a reduced subscheme)* [71]*, locally a complete intersection in M.*

(b) *$M \smallsetminus H$ is rationally convex* ([Mok2])*.*

Proof under assumption (a). For every $x \in H$, by assumption there exist a polynomial $P \in \mathbb{C}[z_1, \ldots, z_N]$ and a Zariski neighborhood $V(x) \subset M$ such that $H \cap V(x) = P^{-1}(0) \cap V(x)$. Let H' be the union of the irreducible components of $P^{-1}(0)$ not contained in H. Since $x \notin H'$, there exists a polynomial Q vanishing on H' such that $Q(x) = 1$. Hilbert's Nullstellensatz implies the existence of an integer $s \in \mathbb{N}$ such that $Q^s/P \in R(M \smallsetminus H)$. Since the Zariski topology is quasi-compact, we may extract a finite covering $V(x_1), \ldots, V(x_m)$ of H and a finite family of polynomials P_j, $Q_j^{s_j}$ associated with the points x_j. Our construction then shows that the morphism

$$(z_1, \ldots, z_N, Q_1^{s_1}/P_1, \ldots, Q_m^{s_m}/P_m) : M \smallsetminus H \to \mathbb{C}^{N+m}$$

is a proper embedding.

Proof under assumption (b). We first construct, by descending induction on k, a sequence of closed algebraic subvarieties $M_k \subset M$ of pure dimension k, such that $M_k \cap H$ is a hypersurface of M_k. Set $M_n = M$; if M_k has been constructed, choose a polynomial $P_k \in R(M)$ vanishing on H but not vanishing identically on any irreducible component of M_k; then let M_{k-1} be the union of the irreducible components of $M_k \cap P_k^{-1}(0)$ not contained in H.

We now argue by ascending induction on k. We shall construct rational functions $g_1, \ldots, g_{m_k}$ in $R(M \smallsetminus H)$ such that the morphism

$$\Phi_k : (z_1, \ldots, z_N) \mapsto (z_1, \ldots, z_N, g_1(z), \ldots, g_{m_k}(z)) : M \smallsetminus H \to \mathbb{C}^{N+m_k}$$

[71] **[W]** The phrase "as a reduced subscheme" appears only in the later web TeX and has been kept because it makes the hypothesis precise.

is a proper embedding when restricted to $M_k \smallsetminus H$ (for $k = n$, the theorem will thus be proved). If $k = \dim M_k = 1$, this property is clear, because assumption (b) and the maximum principle imply the existence of rational functions $g_1, \ldots, g_{m_1} \in R(M \smallsetminus H)$ whose restrictions to M_1 have poles at the distinct points $x_1, \ldots, x_{m_1}$ of $M_1 \cap H$. Now suppose that Φ_k has been constructed. Let $\pi_k : \mathbb{C}^{N+m_k} \to \mathbb{C}^N$ be the projection, let $\overline{M}$ be the closure of $\Phi_k(M \smallsetminus H)$ in $\mathbb{C}^{N+m_k}$, and let $\overline{H} = \overline{M} \cap \pi_k^{-1}(H)$, so that

$$\Phi_k : M \smallsetminus H \to \overline{M} \smallsetminus \overline{H}$$

is an isomorphism (with inverse $\Phi_k^{-1} = \pi_{k|\overline{M} \smallsetminus \overline{H}}$). By the induction hypothesis, [72] $\overline{M}_k = \Phi_k(M_k \smallsetminus H)$ is a closed algebraic subvariety of $\overline{M}_{k+1} = \overline{\Phi_k(M_{k+1} \smallsetminus H)}$. Since $\overline{M}_{k+1} \cap (P_{k+1} \circ \pi_k)^{-1}(0)$ is the disjoint union $(\overline{M}_{k+1} \cap \overline{H}) \cup \overline{M}_k$, we see that $\overline{M}_{k+1} \cap \overline{H}$ is locally a complete intersection in $\overline{M}_{k+1}$ (and is locally defined there by $P_{k+1} \circ \pi_k$). For every $x \in \overline{M}_{k+1} \cap \overline{H}$, there exists a polynomial $Q \in R(\overline{M})$ such that $Q(x) = 1$ and which vanishes on all irreducible components of $(P_{k+1} \circ \pi_k)^{-1}(0)$ that do not meet $\overline{M}_{k+1} \cap \overline{H}$. We may therefore complete Φ_k to a morphism Φ_{k+1} as in case (a), by adjoining to Φ_k the functions $g_j = (Q_j \circ \Phi_k)^{s_j}/P_{k+1}$, $m_k < j \leqslant m_{k+1}$; then $\Phi_{k+1} : M_{k+1} \smallsetminus H \to \mathbb{C}^{N+m_{k+1}}$ is proper, because the functions $g_j \circ \pi_k = Q_j^{s_j}/(P_{k+1} \circ \pi_k)$ define a proper morphism on $\overline{M}_{k+1} \smallsetminus \overline{H}$. □

Property 13.3 (a) implies that $K_\varphi(X)$ is generated by $f_1, \ldots, f_N$, and hence that $K_\varphi(X)$ is also the field of fractions of $A^0_\varphi(X)$, which was by no means obvious a priori. In fact, we can obtain a slightly more precise result.

Proposition 13.5. – *Under assumption* 9.1'(b') [*resp.* 9.1 (b)], $K_\varphi(X)$ *is generated by* $A^b_\varphi(X)$, *where* $b = \frac{2c}{1+c}$ [*resp. by* $A^2_\varphi(X)$].

Proof. In view of the preceding argument, it suffices to construct an injective embedding

$$G = (g_1, \ldots, g_s) : X \to \mathbb{C}^s, \qquad g_j \in A^b_\varphi(X).$$

Proposition 11.5 (c) makes it possible, for all points $x_0 \in X$ and $(x_1, x_2) \in X \times X \smallsetminus \Delta$ (where $\Delta =$ the diagonal), to construct functions $g_1, \ldots, g_n, g \in A^b_\varphi(X)$ such that $dg_1 \wedge \ldots \wedge dg_n(x_0) \neq 0$ and $g(x_1) \neq g(x_2)$. Since the open sets

$$\{x\,;\ dg_1 \wedge \ldots \wedge dg_n(x) \neq 0\} \quad \text{and} \quad \{(x,y)\,;\ g(x) \neq g(y)\} \subset X \times X \smallsetminus \Delta$$

are Zariski open, there exist finite coverings of X and $X \times X \smallsetminus \Delta$, respectively, by such open sets. The collection of functions g, g_j thus obtained gives the desired morphism G. □

[72] **[E]** In this ascending induction step, the polynomial is P_{k+1} throughout. The new indices satisfy $m_k < j \leqslant m_{k+1}$, so the last index is included.

Remark 13.6. – In full generality we have the inclusions

$$A^\infty_\varphi(X) \subset A^b_\varphi(X) \subset A^0_\varphi(X) \subset A_\varphi(X), \qquad 0 < b \leqslant 2,$$

but for the last two we do not know whether equality always holds. The surprising fact is that the algebra $A^\infty_\varphi(X)$ can have transcendence degree $< n$. For example, choose $X = \mathbb{C}$, with the strictly psh function

$$\varphi(z) = \sum_{j\in\mathbb{N}} 2^{-j} \log(\varepsilon_j + |z-j|^2), \qquad 0 < \varepsilon_j \leqslant 1, \quad \varepsilon_0 = 1.$$

Given $r \geqslant 3$, splitting the sum into the indices $j \leqslant \log_2 r$ on the one hand and $j > \log_2 r$ on the other, we readily obtain for $|z| = r$ the estimates

$$\text{(13.7)} \quad \varphi(z) = 2\log(1+|z|^2) + O\Big(\frac{\log r}{r}\Big) \qquad \text{if } \forall j,\ |z-j| > \tfrac{1}{2},$$

$$\text{(13.8)} \quad \varphi(z) = 2\log(1+|z|^2) + 2^{-j}\log(\varepsilon_j + |z-j|^2) + O\Big(\frac{\log r}{r}\Big) \qquad \text{if } |z-j| \leqslant \tfrac{1}{2}.$$

Choose ε_j so that

$$\text{(13.9)} \quad 2\log(1+j^2)+2^{-j}\log\varepsilon_j = \log(1+\log j), \quad j \geqslant 1, \quad \text{i.e. } \varepsilon_j = \Big[\frac{1+\log j}{(1+j^2)^2}\Big]^{2^j}.$$

We then have $\varphi(j) \sim \log\log j$ as $j \to +\infty$, so φ is an exhaustion function. The functions $f \in A^\infty_\varphi(\mathbb{C})$ have polynomial growth and must in addition satisfy $|f(j)| \leqslant (\log j)^{\mathrm{const}}$ as $j \to +\infty$. Consequently $A^\infty_\varphi(\mathbb{C})$ consists only of the constants. Conditions 9.1 (a) and (b) nevertheless hold. Indeed, a direct computation gives

$$dd^c\varphi = 2i\, dz \wedge d\overline{z} \sum_{j\in\mathbb{N}} \frac{2^{-j}\varepsilon_j}{(\varepsilon_j + |z-j|^2)^2},$$

so that $\int_{\mathbb{C}} dd^c\varphi = 8\pi$. The upper bound 9.1 (b) holds with the function

$$\psi(z) = -\log\Bigg(\sum_{j\in\mathbb{N}} \frac{2^{-j}\varepsilon_j}{(\varepsilon_j + |z-j|^2)^2}\Bigg).$$

Considering only the term $j=0$, we obtain the upper bound $\psi(z) \leqslant 2\log(1+|z|^2)$. For $\varepsilon_j^{1/3} \leqslant |z-j| \leqslant \frac{1}{2}$, on the other hand, (13.8) and (13.9) $\times\ \frac{2}{3}$ give:

$$\varphi(z) \geqslant 2\log(1+|z|^2) + 2^{-j}\frac{2}{3}\log\varepsilon_j + O(1) \geqslant \frac{2}{3}\log(1+|z|^2) + O(1),$$

whereas for $|z-j| \leqslant \varepsilon_j^{1/3}$ we obtain

$$\frac{2^{-j}\varepsilon_j}{(\varepsilon_j + |z-j|^2)^2} \geqslant \frac{2^{-j}\varepsilon_j}{4\,\varepsilon_j^{4/3}} = 2^{-j-2}\varepsilon_j^{-1/3},$$

so that $\psi(z) \leqslant 0$ if j is sufficiently large. We thus see that there exists a constant B such that $\psi \leqslant 3\varphi + B$. □

14 Algebraicity of singular complex spaces.

Let X be an analytic space of dimension n. If X is an algebraic set in $\mathbb{C}^N$, the computations of §10 show that the geometric conditions 9.1 (a,b,c) are satisfied.

Conversely, in proving the sufficiency of the geometric conditions, we encounter two main difficulties. First, Hörmander's L^2 estimates are a priori valid only on a smooth Stein open set of the form $X \smallsetminus H$, where H is a hypersurface of X containing the singular locus X_{sing}. In order to apply an extension lemma, we must therefore assume that X is normal.

Lemma 14.1. – *Let f be a holomorphic function on $X \smallsetminus H$ such that $f \in L^2_{\text{loc}}(X_{\text{reg}})$. Then, if X is normal, f extends to a holomorphic function on X.*

Proof. Under the assumption $f \in L^2_{\text{loc}}(X_{\text{reg}})$, it is classical that f extends from $X_{\text{reg}} \smallsetminus H$ to X_{reg}, and every holomorphic function on X_{reg} extends to X if X is normal (cf. [Nar], Proposition VI.4). □

Another difficulty arises from the fact that the weight $e^{-\psi}$ need not be locally integrable at certain singular points with respect to a smooth ambient metric. Consider, for example, the case of the conical variety X with equation $z_0^p + \cdots + z_n^p = 0$ in $\mathbb{C}^{n+1}$, $p \in \mathbb{N}^*$. The Ricci curvature of X is then given, by Proposition 10.1 (a), by the formula

$$\text{Ricci}(\beta_{|X}) = -\frac{1}{2} dd^c\psi$$

with $\psi(z) = \log(|z_0|^{2p-2} + \cdots + |z_n|^{2p-2})$. Thus the function $e^{-\psi}$ is locally integrable at 0 only if $p \leqslant n$. In the case of a space X for which $e^{-\psi}$ is not integrable near every point of a curve, Proposition 11.5 (c) no longer applies. We are therefore led to assume that the singularities of X are isolated.

Proof of Theorem 9.1' (sufficiency of the conditions in the case of isolated singularities). Assumption (c') implies that the irreducible components of X are finite in number. Let

$$\pi : \tilde{X} \to X$$

be the normalization of X. The function $\varphi \circ \pi$ is not in general strictly psh near $\pi^{-1}(X_{\text{sing}})$, but after modifying $\varphi \circ \pi$ and $\psi \circ \pi$ near the finite set $\pi^{-1}(X_{\text{sing}})$, we see that the assumptions are satisfied by $\tilde{X}$. Ultimately, we may assume that X is normal and irreducible.

The proof is now entirely similar to the one given in §11, 12, 13, so we shall merely indicate the main lines and the necessary changes. Lemmas 11.2 and 11.3 hold without any modification, as do Properties 11.5 (a,b,d). Assertion 11.5 (c) remains valid if $\{x_1, \ldots, x_m\} \subset X_{\text{reg}}$, and if some of the points x_j are singular, we have the following partial result (corresponding to the case $\rho = 0$).

Lemma 14.2. – *Let $\{x_1, \ldots, x_m\} \subset X$ be a finite set. Then there exists a function $f \in A^b_\varphi(X)$, $b = \frac{2c}{1+c}$, having any prescribed jet of order s at each point $x_1, x_2, \ldots, x_m$.*

Proof. We use the same arguments as in 11.5 (c), replacing the system of local coordinates $z^{(j)}$ by a generating set $(z_1^{(j)}, \ldots, z_{N_j}^{(j)})$ of the maximal ideal $\mathfrak{m}_{X,x_j}$ of $\mathcal{O}_{X,x_j}$, and ρ_1 by [73]

$$\rho_1 = (s+1)(n+2)\left[\sum_{j=1}^{m} \chi_j \log|z^{(j)}|^2 + C_1\varphi\right].$$

Near x_j, the construction then gives $f = P_j(z^{(j)}) - g$ with g holomorphic and such that [74]

$$|g|^2 \; |z^{(j)}|^{-2(s+1)(n+2)} \; e^{-\psi} \in L^1_{\mathrm{loc}}(X).$$

By H. Skoda's L^2 estimates [Sk4], this implies that $g \in \mathfrak{m}^{s+1}_{X,x_j}$. □

Proposition 11.7 therefore remains applicable if $x_0 \in X_{\mathrm{reg}}$, and by repeating the arguments of §12, 13, we construct a normal algebraic variety M and a morphism $F = (f_1, \ldots, f_N) : X \to M$ whose restriction to X_{reg} is an isomorphism from X_{reg} onto a Zariski-open subset of M. By Lemma 14.2, we may (after supplementing F by finitely many functions f_j) assume that F defines an embedding of X near every singular point. The morphism F is then an isomorphism from X onto the Zariski-open subset $F(X) \subset M$. The end of the proof is identical to that given in §13. □

The argument just outlined also yields the following interesting result.

Theorem 14.3. – *Let X be a normal analytic space of dimension n, satisfying assumptions* 9.1'(a',b',c'). *Then X_{reg} is analytically isomorphic to a quasi-affine algebraic variety, the isomorphism being given by a φ-polynomial morphism F from X into a normal affine algebraic variety $M \subset \mathbb{C}^N$ of dimension n.* □

---

[73] **[E]** To prescribe a jet of order s, the construction must force the correction term into the ideal power $\mathfrak{m}^{s+1}$. The weight therefore also uses $s+1$.

[74] **[E]** The displayed integrand already contains $|g|^2$. Its correct local integrability class is therefore L^1_{loc}, not L^2_{loc}.

15 Appendix: currents and plurisubharmonic functions of minimal growth on an affine algebraic variety.

Let M be a smooth affine algebraic subvariety of dimension n. We equip M with the Kähler metrics

$$\beta = dd^c|z|^2, \qquad \omega = dd^c\log(1+|z|^2)$$

induced respectively by the flat metric of $\mathbb{C}^N$ and the Fubini–Study metric of projective space $\mathbb{P}^N$.

Definition 15.1. – *Psh functions and currents of minimal growth:*

(a) *A psh function V on M is said to have minimal growth if there exist constants $C_0, C_1 \geqslant 0$ such that*

$$V(z) \leqslant C_1 \log_+|z| + C_0.$$

(b) *A closed positive current T of bidegree $(1,1)$ on M is said to have minimal growth if*

$$\int_M T\wedge\omega^{n-1} < +\infty.$$

Corollary 7.3 immediately gives

Proposition 15.2. – *If V is psh and has minimal growth on M, then $T = dd^cV$ has minimal growth.* □

Conversely, given a closed current $T \geqslant 0$ of minimal growth, we can find a solution of the equation $dd^cV = T$ only if the cohomology class of T vanishes. The purpose of this section is to prove the following general result, which is a partial converse to Proposition 15.2.

Theorem 15.3. – *Let T be a closed positive $(1,1)$-current on M such that*

$$\int_M T\wedge\omega^{n-1} < +\infty.$$

Then there exist a psh function V and a $(1,0)$-form u of class $\mathcal{C}^\infty$ on M having the properties below, where C_1, C_2, C_3 are constants $\geqslant 0$.

(a) $dd^cV \geqslant T$;

(b) $V(z) \leqslant C_1 \log_+|z|$;

(c) $dd^cV - T = \overline{\partial}u$;

(d) $|u|_\omega \leqslant C_2(1+|z|)^{C_3}$.

The proof will proceed in several steps. First observe that condition 15.1 (b) is equivalent to the following:

$$\sigma(r) = \int_{|\zeta|<r} T(\zeta)\wedge\beta^{n-1} \leqslant C\, r^{2n-2}. \tag{15.4}$$

There is no loss of generality in assuming $n \geqslant 2$. Otherwise, we may apply Theorem 15.3 to the variety $M' = M\times\mathbb{C}$ and to the pullback current $T' = \pi_M^* T$. The function $V(z) = V'(z,0)$ and the form $u = u'_{|M\times\{0\}}$ then have the required properties.

Given a current $T \geqslant 0$ of bidegree $(1,1)$ on M satisfying (15.4), we may associate with it a potential V_T by the same formulas as those used by P. Lelong [Le3] in $\mathbb{C}^n$:

$$V_T(z) = \int_M T(\zeta)\wedge\beta^{n-1}\, L_n(z,\zeta) \tag{15.5}$$

with

$$L_n(z,\zeta) = \frac{1}{(n-1)(4\pi)^n}\left[\frac{1}{(1+|\zeta|^2)^{n-1}} - \frac{1}{|z-\zeta|^{2n-2}}\right].$$

Lemma 15.6. – *Formula* (15.5) *defines a function $V_T \in L^1_{\mathrm{loc}}(M)$ that is upper semicontinuous. Moreover, there exist constants C_0, C_1 such that*

$$V_T(z) \leqslant C_1 \log_+|z| + C_0.$$

Proof. The kernel L_n clearly satisfies the following estimates:

$$|L_n(z,\zeta)| \leqslant C_2|z|\,|\zeta|^{1-2n} \qquad \text{if } |\zeta| \geqslant 2|z| \geqslant 1,$$

$$L_n(z,\zeta) \leqslant C_3|\zeta|^{2-2n} \qquad \text{if } 1 \leqslant |\zeta| \leqslant 2|z|.$$

For $|z| = r \geqslant 1$, we deduce

$$V_T(z) \leqslant C_4\left[1 + \int_1^{2r}\frac{1}{t^{2n-2}}\,d\sigma(t) + \int_{2r}^{+\infty}\frac{r}{t^{2n-1}}\,d\sigma(t)\right].$$

After integration by parts, in view of (15.4), we obtain:

$$\begin{aligned} V_T(z) &\leqslant C_4\left[1 + \frac{\sigma(2r)}{(2r)^{2n-2}} + (2n-2)\int_1^{2r}\frac{\sigma(t)\,dt}{t^{2n-1}} + (2n-1)r\int_{2r}^{+\infty}\frac{\sigma(t)\,dt}{t^{2n}}\right] \\ &\leqslant C_5(1+\log r). \end{aligned}$$

The preceding estimates also show that the integral (15.5) converges absolutely on the set $\{\zeta \in M\,;\ |\zeta| > 2|z|\}$, uniformly when z ranges over a compact subset of M. The property $V_T \in L^1_{\rm loc}(M)$ then follows from Fubini's theorem and the fact that $|L_n(z,\zeta)|$ is, locally on $M \times M$, integrable in z uniformly with respect to ζ. □

Lemma 15.7. – *For every point $z \in M$, there exist balls $B'_z \subset T_zM$, $B''_z \subset (T_zM)^\perp$ centered at 0 and of radius $r(z) = C_6(1+|z|)^{-C_7}$, where $C_6, C_7 > 0$, and a holomorphic map $g_z : B'_z \to B''_z$ such that $M \cap (z + B'_z + B''_z)$ is the graph of g_z, i.e. if $\zeta - z = \zeta' + \zeta''$ is the decomposition of a point $\zeta \in M$ according to the decomposition $\mathbb{C}^N = (T_zM) \oplus (T_zM)^\perp$, then*

$$M \cap (z + B'_z + B''_z) = \{\zeta \in \mathbb{C}^N\,;\ \zeta'' = g_z(\zeta'),\ \zeta' \in B'_z\}.$$

Proof. Let $(P_1, \ldots, P_m)$ be a system of polynomial generators for the ideal of the variety M in $\mathbb{C}[X_1, \ldots, X_N]$. Since M is smooth, the partial Jacobians $J_{K,L}$ of order $N - n$ (cf. §10) do not all vanish simultaneously on M. By Hilbert's Nullstellensatz, the polynomials $J_{K,L}$ generate the unit ideal on M; hence there exist constants $C_8, C_9 > 0$ such that

$$\max_{K,L} |J_{K,L}(z)| \geqslant C_8(1+|z|)^{-C_9}, \qquad z \in M.$$

The lemma then follows from the implicit function theorem (in its quantitative version). □

We now observe that formula (15.5) can be rewritten in the form

$$V_T(z) = \int_M T(\zeta) \wedge \big[K_n(z,\zeta) - H_n(\zeta)\big] \tag{15.8}$$

with

$$K_n(z,\zeta) = -\frac{1}{(n-1)(4\pi)^n}\left[\frac{dd^c|z-\zeta|^2}{|z-\zeta|^2}\right]^{n-1},$$

$$H_n(\zeta) = -\frac{1}{(n-1)(4\pi)^n}\,\frac{\beta(\zeta)^{n-1}}{(1+|\zeta|^2)^{n-1}}.$$

The properties of the kernel K_n will allow us to compute dd^cV_T easily in terms of T.

Lemma 15.9. – *$dd^cK_n = [\Delta] + R_n$, where $[\Delta]$ is the current of integration over the diagonal of $M \times M$ and R_n is an (n,n)-current $\geqslant 0$ with locally integrable coefficients on $M \times M$, satisfying the estimate*

$$\|R_n(z,\zeta)\|_{\beta\oplus\beta} \leqslant C_{10}\min\left[\frac{1}{|z-\zeta|^{2n}}\,,\ \frac{(1+|z|)^{C_{11}}}{|z-\zeta|^{2n-1}}\right].$$

Proof. Away from the diagonal Δ, a classical computation (whose verification is left to the reader) gives

$$
\begin{aligned}
dd^c K_n &= \frac{(dd^c|z-\zeta|^2)^n - n\,|z-\zeta|^{-2}\,d|z-\zeta|^2\wedge d^c|z-\zeta|^2\wedge(dd^c|z-\zeta|^2)^{n-1}}{(4\pi)^n\,|z-\zeta|^{2n}}\\
&= \Big(\frac{1}{4\pi}\,dd^c\log|z-\zeta|^2\Big)^n, \qquad (15.10)
\end{aligned}
$$

and we shall see below that $\mathbf{1}_\Delta\,dd^cK_n = [\Delta]$. In particular, we have

$$R_n = \mathbf{1}_{M\times M\smallsetminus\Delta}\,dd^cK_n \geqslant 0 \quad\text{and}\quad \|R_n(z,\zeta)\| \leqslant C|z-\zeta|^{-2n}.$$

To obtain the second part of the upper bound, fix a point $z\in M$ and use Lemma 15.7. On restricting to M, at the point z we have

$$dz = dz' = \text{component of } dz \text{ on } T_zM,$$

whereas at a nearby point $\zeta\in z+(B'_z+B''_z)$ we have:

$$d\zeta = d\zeta' + d(g_z(\zeta')).$$

It follows from (15.10): [75]

$$
\begin{aligned}
R_n(z,\zeta) = \Bigg[&\frac{dd^c\big(|z'-\zeta'|^2+|g_z(\zeta')|^2\big)}{(4\pi)\,\big(|\zeta'|^2+|g_z(\zeta')|^2\big)}\\
&\qquad - \frac{d\big(|z'-\zeta'|^2+|g_z(\zeta')|^2\big)\wedge d^c\big(|z'-\zeta'|^2+|g_z(\zeta')|^2\big)}{(4\pi)\,\big(|\zeta'|^2+|g_z(\zeta')|^2\big)^2}\Bigg]^n,
\end{aligned}
$$

where the differentiation of $g_z(\zeta')$ acts only on ζ'. By construction, Lemma 15.7 gives $g_z(0) = D_0g_z = 0$; Schwarz's lemma then implies the inequalities

$$
\begin{aligned}
&|g_z(\zeta')| \leqslant |\zeta'|, && \zeta'\in B'_z;\\
&\|D_{\zeta'}g_z(\zeta')\| \leqslant C(\lambda)\frac{|\zeta'|}{r(z)}, && \zeta'\in\lambda B'_z,\quad 0<\lambda<1.
\end{aligned}
$$

We now observe that $R_n(z,\zeta)\equiv 0$ if $g_z\equiv 0$. It follows that, for $\zeta'\in\frac12 B'_z$,

$$\|R_n(z,\zeta)\| \leqslant \frac{C_{12}\,|\zeta'|\,r(z)^{-1}}{\big(|\zeta'|^2+|g_z(\zeta')|^2\big)^n} \leqslant \frac{C_{12}\,r(z)^{-1}}{|z-\zeta|^{2n-1}},$$

[75] **[C]** The two coordinate expressions look different because the calculation takes place on $M\times M$: the differentials involve both variables, while the denominators are evaluated at the centered point $z'=0$ in the chart of Lemma 15.7. The formulas have therefore been left unchanged.

which completes the estimate in Lemma 15.9. The classical Bochner–Martinelli formula in $\mathbb{C}^n$, on the other hand, gives

$$dd^c K_n(z', \zeta') = [\Delta].$$

By a computation analogous to the one above, we obtain the inequality

$$\|K_n(z,\zeta) - K_n(z',\zeta')\| \leqslant \frac{C_{13}\, r(z)^{-1}}{|z-\zeta|^{2n-3}},$$

and for each differentiation of K_n the exponent of $|z-\zeta|$ increases by one. Thus $dd^c K_n - [\Delta]$ has coefficients in L^1_{loc} on $M \times M$, and consequently carries no mass on Δ. The proof is complete. $\square$

Proposition 15.11. – *If T is closed, then*

$$dd^c V_T = T + \Theta_T \qquad \textit{where} \quad \Theta_T(z) = \int_M R_n(z,\zeta) \wedge T(\zeta) \geqslant 0.$$

In particular, V_T is psh.

Proof. Let $\chi : \mathbb{R} \to [0,1]$ be a function of class $\mathcal{C}^\infty$ such that $\chi(t) = 1$ for $t < 1$, $\chi(t) = 0$ for $t > 2$, and let w be an $(n-1, n-1)$-form of class $\mathcal{C}^\infty$ with compact support on M. Expression (15.8) gives

$$\int_M V_T\, dd^c w = \lim_{r\to+\infty} I(r),$$

$$I(r) = \int_{M\times M} \chi\Big(\frac{|\zeta|}{r}\Big)\, T(\zeta) \wedge \big(K_n(z,\zeta) - H_n(\zeta)\big) \wedge dd^c w(z).$$

Stokes' theorem and Lemma 15.9 imply

$$\begin{aligned}
I(r) &= \int_{M\times M} dd^c\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\, T(\zeta) \wedge K_n(z,\zeta)\Big] \wedge w(z)\\
&= \int_{M\times M} \chi\Big(\frac{|\zeta|}{r}\Big)\, T(\zeta) \wedge \big([\Delta] + R_n(z,\zeta)\big) \wedge w(z)\\
&\qquad + \int_{M\times M} d\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\Big] \wedge T(\zeta) \wedge d^c K_n(z,\zeta) \wedge w(z)\\
&\qquad - \int_{M\times M} d^c\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\Big] \wedge T(\zeta) \wedge dK_n(z,\zeta) \wedge w(z)\\
&\qquad + \int_{M\times M} dd^c\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\Big] \wedge T(\zeta) \wedge K_n(z,\zeta) \wedge w(z)
\end{aligned}$$

because $dT = d^cT = 0$. To justify this computation, we may first assume that T is of class $\mathcal{C}^\infty$, and then regularize T near the support of $\chi(\frac{|\zeta|}{r}) \subset \{|\zeta| \leqslant 2r\}$. We now use (15.4) and the evident estimates [76]

$$\Big\|d\chi\Big(\frac{|\zeta|}{r}\Big)\Big\| = O\Big(\frac{1}{r}\Big), \qquad \Big\|dd^c\chi\Big(\frac{|\zeta|}{r}\Big)\Big\| = O\Big(\frac{1}{r^2}\Big),$$
$$\|d^cK_n(z,\zeta)\| = O\Big(\frac{1}{|z-\zeta|^{2n-1}}\Big), \qquad \|K_n(z,\zeta)\| = O\Big(\frac{1}{|z-\zeta|^{2n-2}}\Big), \quad n \neq 1$$

to see that the three error integrals in the computation of $I(r)$ [77] admit an upper bound of the form $O(r^{-2})$. We therefore obtain the expected formula

$$\lim_{r\to+\infty} I(r) = \int_M T(\zeta)\wedge w(\zeta) + \int_{M\times M} T(\zeta)\wedge R_n(z,\zeta)\wedge w(z). \qquad \square$$

Proof of Theorem 15.3. By Proposition 15.2 and Lemma 15.6, the current Θ_T is closed, positive, and of minimal growth. We may therefore construct, by induction on k, psh functions V_k and closed positive currents T_k of minimal growth such that

$$T_0 = T, \qquad V_k = V_{T_{k-1}}, \qquad T_k = \Theta_{T_{k-1}},$$
$$dd^cV_k = T_{k-1} + T_k.$$

Take the alternating sum of these identities. For odd indices we obtain:

$$dd^c\big(V_1 - V_2 + \cdots - V_{2k} + V_{2k+1}\big) = T + T_{2k+1} \geqslant 0,$$

and Lemma 15.15 below implies that the psh function $V = V_1 - V_2 + \cdots + V_{2k+1}$ has minimal growth. By Proposition 15.11, we have the recurrence relation

$$T_{k+1}(z) = \int_M R_n(z,\zeta)\wedge T_k(\zeta).$$

We now exploit the fact that R_n is a smoothing kernel of convolution type.

Lemma 15.12. – *We have the following properties.*

(a) *For every integer k with $1 \leqslant k < 2n$, there exist constants $A_k, B_k \geqslant 0$ such that, for every $\varepsilon \in \,]0,1[$, one has*

$$\|T_k(z)\| \leqslant A_k(1+|z|)^{B_k}\Big[\varepsilon^{-2} + \int_{|\zeta-z|<\varepsilon\, r(z)} \frac{T(\zeta)\wedge\beta(\zeta)^{n-1}}{|\zeta-z|^{2n-k}}\Big],$$

where $r(z) = C_6(1+|z|)^{-C_7}$ [cf. Lemma 15.7].

[76] **[P]** The last estimate concerns K_n, as in the published paper. The web TeX incorrectly replaces it by dd^cK_n.

[77] **[E]** The integration-by-parts formula above has three error integrals, all of which must be estimated. The source text mentions only two.

(b) *For $k \geqslant 3$ the current T_k has continuous coefficients and*

$$\|T_k(z)\| = O\big((1+|z|)^{B_k}\big).$$

Proof.

(a) We argue by induction on k. Set

$$\sigma_k(z,r) = \int_{|\zeta - z|<r} T_k(\zeta)\wedge\beta(\zeta)^{n-1}.$$

We know that the function $r \mapsto r^{2-2n}\sigma_k(z,r)$ is increasing and that it tends to $\int_M T_k\wedge\omega^{n-1} < +\infty$ as $r\to+\infty$. Write $T_{k+1}(z) = I_1(z) + I_2(z)$, where

$$I_1(z) = \int_{|\zeta-z|\geqslant\varepsilon\, r(z)} R_n(z,\zeta)\wedge T_k(\zeta),$$
$$I_2(z) = \int_{|\zeta-z|<\varepsilon\, r(z)} R_n(z,\zeta)\wedge T_k(\zeta).$$

We now use Lemma 15.9 to estimate $I_1(z)$ and $I_2(z)$. Up to a constant, the norm $\|I_1(z)\|$ is bounded above by

$$\begin{aligned}\int_{\varepsilon\, r(z)}^{+\infty}\frac{d\sigma_k(z,r)}{r^{2n}} &\leqslant 2n\int_{\varepsilon\, r(z)}^{+\infty}\frac{\sigma_k(z,r)}{r^{2n+1}}\,dr\\ &\leqslant \frac{n}{\varepsilon^2 r(z)^2}\int_M T_k\wedge\omega^{n-1} = O\big(\varepsilon^{-2}(1+|z|)^{2C_7}\big),\end{aligned}$$

whereas

$$\|I_2(z)\| \leqslant C_{10}(1+|z|)^{C_{11}}\int_{|\zeta-z|<\varepsilon\, r(z)}\frac{\|T_k(\zeta)\|\;\beta(\zeta)^n}{|\zeta-z|^{2n-1}}. \tag{15.13}$$

When $k=0$, this proves estimate (a) for $\|T_1(z)\|$. In the general case, the estimate of order k, combined with (15.13), gives

$$\|I_2(z)\| \leqslant C_{14}(1+|z|)^{B_k+C_{11}}\big(\varepsilon^{-2}I_3(z)+I_4(z)\big)$$

with

$$I_3(z) = \int_{|\zeta-z|<\varepsilon\, r(z)}\frac{\beta(\zeta)^n}{|\zeta-z|^{2n-1}},$$
$$I_4(z) = \int_{|\zeta-z|<\varepsilon\, r(z)}\frac{\beta(\zeta)^n}{|\zeta-z|^{2n-1}}\int_{|w-\zeta|<\varepsilon\, r(\zeta)}\frac{T(w)\wedge\beta(w)^{n-1}}{|w-\zeta|^{2n-k}}.$$

For ε sufficiently small, the inequalities $|\zeta-z|<\varepsilon\, r(z)$ and $|w-\zeta|<\varepsilon\, r(\zeta)$ imply $|w-z|<3\varepsilon\, r(z)$. With the notation of Lemma 15.7, the integrals $I_3(z)$ and $I_4(z)$ therefore admit the upper bounds

$$I_3(z) \leqslant C_{15}\int_{|\zeta'|<\varepsilon\, r(z)}\frac{\beta(\zeta')^n}{|\zeta'|^{2n-1}} \leqslant C_{16}\;\varepsilon\, r(z),$$

$$I_4(z) \leqslant C_{17}\int_{|w-z|<3\varepsilon\, r(z)} T(w)\wedge\beta(w)^{n-1}\int_{\zeta'\in\mathbb{C}^n}\frac{\beta(\zeta')^n}{|\zeta'|^{2n-1}|w'-\zeta'|^{2n-k}}.$$

By homogeneity, we obtain

$$\int_{\zeta'\in\mathbb{C}^n} \frac{\beta(\zeta')^n}{|\zeta'|^{2n-1}|w'-\zeta'|^{2n-k}} = \frac{C_{18}}{|w'|^{2n-k-1}} \leqslant \frac{C_{19}}{|w-z|^{2n-k-1}},$$

and estimate (a) follows at order $k+1$.

(b) Use inequality (a) for $k \geqslant 3$. We obtain [78]

$$\begin{aligned}\int_{|\zeta-z|<\varepsilon\, r(z)} \frac{T(\zeta)\wedge\beta(\zeta)^{n-1}}{|z-\zeta|^{2n-k}} &= \int_0^{\varepsilon\, r(z)} \frac{d\sigma_0(z,r)}{r^{2n-k}} \\ &= \frac{\sigma_0(z,\varepsilon\, r(z))}{[\varepsilon\, r(z)]^{2n-k}} + (2n-k)\int_0^{\varepsilon\, r(z)} \frac{\sigma_0(z,r)\; dr}{r^{2n-k+1}} \\ &= O\big((\varepsilon\, r(z))^{k-2}\big).\end{aligned}$$

Estimate (b) follows. Moreover, the preceding integral converges uniformly to 0 as $\varepsilon\to 0$. In estimate (a), this integral corresponds to the iteration of the kernel R_n over the balls $|\zeta - z| < \varepsilon\, r(z)$. All the other terms contributing to T_k involve at least one integration over the complement $\{|\zeta - z| \geqslant \varepsilon\, r(z)\}$ and are therefore continuous in z. Thus T_k is continuous as soon as $k\geqslant 3$. □

Proof of Theorem 15.3 (continued). At this point, we have constructed a psh function V of minimal growth and a closed positive current Θ with continuous coefficients such that

$$dd^cV = T + \Theta, \qquad \|\Theta(z)\| = O\big((1+|z|)^{C_{20}}\big).$$

We begin by showing that we may assume Θ to be of class $\mathcal{C}^\infty$. Let $(\Omega_j, g_j)_{j\in\mathbb{N}}$ be a locally finite atlas of M, with $\Omega_j \Subset M$ and $g_j : \Omega_j \to \mathbb{C}^n$ a biholomorphic map from Ω_j onto the unit ball of $\mathbb{C}^n$, and let $(\psi_j)_{j\in\mathbb{N}}$ be a $\mathcal{C}^\infty$ partition of unity on M subordinate to this atlas. There exist psh functions τ_j on Ω_j such that $dd^c\tau_j = \Theta$. Denote by $\tau_j^\varepsilon = \tau_j * \rho_\varepsilon$ a family of $\mathcal{C}^\infty$ regularizations of τ_j with respect to the chart g_j, and set

$$W = \sum_{j\in\mathbb{N}} \psi_j(\tau_j - \tau_j^{\varepsilon_j}), \qquad \varepsilon_j > 0.$$

On the open set Ω_k we have

$$dd^cW - \Theta = dd^c(W-\tau_k) = dd^c\Big[\sum_{j\in\mathbb{N}} \psi_j(\tau_j - \tau_k - \tau_j^{\varepsilon_j})\Big],$$

and since $\tau_j - \tau_k \in \mathcal{C}^\infty(\Omega_j\cap\Omega_k)$, we see that $dd^cW - \Theta \in \mathcal{C}^\infty_{1,1}(M)$. Since the current Θ has continuous coefficients, τ_j and the 1-forms $d\tau_j$, $d^c\tau_j$ are continuous.

[78] **[P/E]** This Stieltjes calculation follows the published formula and uses σ_0, because the integral contains the original current $T = T_0$.

When the ε_j are chosen sufficiently small, we therefore obtain $|W| \leqslant 1$ and $-\omega \leqslant dd^cW \leqslant \omega$, with $\omega = dd^c \log(1 + |z|^2)$. The function $V' = V - W + \log(1 + |z|^2)$ is then psh and has minimal growth, and satisfies $dd^cV' = T + \Theta'$, where

$$\Theta' = \Theta - dd^cW + \omega$$

is a closed positive current of class $\mathcal{C}^\infty$ such that $\|\Theta'(z)\| = O\big((1 + |z|)^{C_{20}}\big)$.

We henceforth assume that Θ is of class $\mathcal{C}^\infty$. We then apply the Hörmander–Nakano–Skoda L^2 estimates [Nak], [Sk4] to the current Θ, regarded as a closed $(n, 1)$-form with values in the holomorphic bundle $E = T^*M \otimes \bigwedge^n TM$. The cotangent bundle T^*M is semipositive in the sense of Griffiths for the metric β (it is a quotient of the flat bundle $T^*\mathbb{C}^N_{|M}$), so by [DS] the bundle $T^*M \otimes \bigwedge^n T^*M$ is semipositive in the sense of Nakano. Proposition 10.1 (b) shows that the bundle E itself is semipositive in the sense of Nakano for the metric $\beta\, e^{-2\psi}$, where $\psi = \log\big(\sum_{K,L} |J_{K,L}|^2\big)$. By the estimates of [Sk4] applied to the Hermitian bundle

$$\Big(E, \beta \,\exp\big(-2\psi - C_{21} \log(1 + |z|^2)\big)\Big),$$

there exists a form $u \in \mathcal{C}^\infty_{1,0}(M)$ such that $\overline{\partial}u = \Theta$ and

$$\int_M |u|^2_\beta\, (1 + |z|^2)^{-C_{22}}\, \beta^n < +\infty.$$

To complete the proof of Theorem 15.3 and, in particular, of 15.3 (d), it suffices to convert this L^2 estimate into an L^∞ estimate [79]

$$|u|_\beta \leqslant C_{23}(1 + |z|)^{C_{24}}.$$

Since $\overline{\partial}u = \Theta$ has an L^∞ bound of polynomial growth, it suffices to use the inequality below in the balls $|\zeta - z| < \frac{1}{2}\, r(z)$ of Lemma 15.7. □

Lemma 15.14. – *Let v be a function of class $\mathcal{C}^1$ in the ball $B(r) \subset \mathbb{C}^n$.*[80] *Then*

$$|v(0)| \leqslant \left[\frac{n!}{\pi^n r^{2n}} \int_{B(r)} |v(z)|^2\, d\lambda(z)\right]^{1/2} + \frac{4n}{2n+1}\, r \cdot \sup_{B(r)} |\overline{\partial}v|.$$

Proof. Apply the Cauchy formula with remainder to the function $t \mapsto v(tz)$, $z \in B(r)$, $t \in \mathbb{C}$, $|t| < 1$. We obtain

$$v(0) = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\theta}z)\, d\theta - \frac{1}{\pi}\int_{|t|<1} \frac{\overline{\partial}v(tz)\cdot z}{t}\, d\lambda(t),$$
$$|v(0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |v(e^{i\theta}z)|\, d\theta + 2|z| \sup_{B(r)} |\overline{\partial}v|.$$

[79] **[E]** The estimate needed here is the polynomial-growth bound $|u|_\beta \leqslant C_{23}(1 + |z|)^{C_{24}}$. Both source versions instead print a squared norm with a decaying right-hand side, which cannot imply Theorem 15.3(d).

[80] **[W]** The later web TeX corrects the numbering: equation (15.13) is followed by Lemmas 15.14 and 15.15. The printed paper repeats 15.13 and then uses 15.14.

After averaging over $z \in B(r)$, we obtain

$$|v(0)| \leqslant \mathrm{VM}\,[\,|v|\;;\;B(r)] + \frac{4n}{2n+1}\;r\cdot\sup_{B(r)}|\overline{\partial}v|,$$

and

$$\mathrm{VM}\,[\,|v|\;;\;B(r)] \leqslant \mathrm{VM}\,[\,|v|^2\;;\;B(r)]^{1/2}$$

by the Cauchy–Schwarz inequality. □

It remains only to verify the following elementary result, which was used in the course of the proof.

Lemma 15.15. – *Let V_1, V_2 be two psh functions of minimal growth on M. Assume that $V = V_1 - V_2$ is psh. Then V has minimal growth.*

Proof. By Noether's normalization theorem, there exist polynomial functions $f_1,\ldots,f_n$ on M such that $R(M) = \mathbb{C}[z_1,\ldots,z_N]/I(M)$ is an integral algebra over $\mathbb{C}[f_1,\ldots,f_n]$. The morphism $F = (f_1,\ldots,f_n) : M \to \mathbb{C}^n$ is therefore proper and finite, and we have two-sided bounds [81]

$$C_{25}(1+|z|)^{C_{26}} \leqslant 1+|F(z)| \leqslant C_{27}(1+|z|)^{C_{28}}$$

with $C_{25},\ldots,C_{28} > 0$. In view of the evident inequality

$$V \leqslant V_+ \leqslant F^*(F_*V_+)$$

it suffices to show that F_*V_+ has minimal growth in $\mathbb{C}^n$. Since

$$V_+ \leqslant (V_1)_+ + (V_2)_+ - V_2,$$

we deduce for the mean value of F_*V_+ over the ball $B(r) \subset \mathbb{C}^n$ centered at 0 the upper bound

$$\mathrm{VM}\,[F_*V_+\,;\,B(r)] \leqslant \mathrm{VM}\,[F_*(V_1)_+ + F_*(V_2)_+\,;\,B(r)] - \mathrm{VM}\,[F_*V_2\,;\,B(r)].$$

The functions $F_*(V_1)_+$, $F_*(V_2)_+$ are psh and have minimal growth, while the function $r \mapsto \mathrm{VM}[F_*V_2\,;\,B(r)]$ is increasing. Consequently, we obtain an upper bound

$$\mathrm{VM}\,[F_*V_+\,;\,B(r)] \leqslant C_{29}\log_+ r + C_{30},$$

and the lemma follows from the mean-value inequalities

$$F_*V_+(z) \leqslant \mathrm{VM}\,[F_*V_+\,;\,B(z,r)] \leqslant 2^{2n}\;\mathrm{VM}\,[F_*V_+\,;\,B(0,2r)] \leqslant C_{31}\log_+ r + C_{32}$$

with $r := |z|$. □

[81] **[E]** The lower bound is written for $1+|F(z)|$ so that it remains valid when $F(z)=0$.

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[92] **[E]** The text cites Whitney's papers [Wh1] and [Wh2], but neither source bibliography lists them. Both entries have therefore been added.