NOTES ON ISOTHERMAL COORDINATES AND INTEGRABILITY OF ALMOST COMPLEX STRUCTURE

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ABSTRACT. This note is the final report for Math 742 Geometric Analysis at UMD, taught by Professor Daniel Cristofaro-Gardiner in Fall 2023. In this note, we discuss Chern's simplified proof for a result of Korn and Litchenstein which concerns the existence of isothermal coordinate, provided the Riemannian metric is Hölder continuous.

CONTENTS

1. INTRODUCTION

Let $ds^2 = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$ be a Riemannian metric defined on an open chart D of a surface S with local coordinates (x,y), where $EG - F^2, E > 0$.

Definition 1.1. We say a local coordinates (u,v) for S is an **isothermal coordinates** (relative to ds^2) if

$$ds^2 = \lambda(u,v)(du^2 + dv^2), \quad \lambda > 0. \tag{1.1}$$

The main purpose of this note is to present the proof on the existence of isothermal coordinates on on a surface with relatively weak regularity assumption on Riemannian metric ds^2. To formulate the main result, we first recall the notion of Hölder continuity.

Definition 1.2. Let $D \subset \mathbb{R}^2$ be a domain. For $\alpha \in (0,1)$, a function $f : D \to \mathbb{C}$ is said to be **α-Hölder continuous** on Ω , denoted by $f \in C^{0,\alpha}(\overline{D})$, if there exists $C > 0$ such that

$$|f(p) - f(p')| \leqslant C|p - p'|^\alpha, \quad \forall p, p' \in D. \tag{1.2}$$

In this case, we denote $[f]_{\alpha,D} := \sup_{p,p' \in D, p \neq p'} |f(p) - f(p')|/|p - p'|^\alpha$ to be the best constant in (1.2).

We now state our main theorems in the note.

Theorem 1.1 (Korn–Litchenstein)**.** *Assume that $E, F, G \in C^{0,\alpha}(D)$, for some $\alpha \in (0,1)$. Then for any $p \in D$, there exists an isothermal coordinates (u,v) defined on a neighborhood $V \subset D$ of p.*

We briefly describe the history on existence of isothermal coordinates. This was first shown by Lagrange for surface of revolutions (1779) and was later generalized by Gauss (1825) for real-analytic metrics using Cauchy–Kowalesky theorem (basically ODE theory). There are simpler proofs for existence of such coordinates in some special cases, say when S is a minimal surface for example (see Do Carmo). For general surface, it was until Korn and Litchenstein in 1916 used 2-dimensional PDE theory to prove the existence for smooth metrics or even Hölder continuous metrics.

One may wonder whether assumption of Hölder continuity on metrics in Theorem 1.1 can be relaxed. However, Hartman and Winter (1953) demonstrated that there exists a continuous metric which admits no isothermal coordinates. Korn–Litchenstein theorem seems to be optimal.

Fact 1.1 (Hartman–Winter)**.** *Let $\overline{D} = D_r(0)$ be closed disk for some $r < 1$.If $E, F, G \in C^0(\overline{D})$, then in any neighborhood V of $0 \in D$, there does not exist C^1-isothermal coordinates (u, v) on V.*

The note is organized as following. In the next section, following Chern's simplication (1954), we reformulate existence problem of isothermal coordinates using complex notations and present some preliminaries discussion about integrals on complex plane. Next, we present a direct Hölder estimate on Cauchy potential. This is similar to the Schauder perturbative approach for ellipitc operators via potential theory where one needs to first obtain Hölder estimates for derivatives of Newton potentials. Finally, we first state Chern's result on solvability of certain inhomogenous integral equation (cf. Theorem 4.1) and use it to deduce Theorem 1.1. The remaining section is devoted to the proof of Theorem 4.1 by utilizing uniform and Hölder estimate established earlier.

We now end this introductory section by discussing some generalization of Theorem 1.1. It is well-known that the existence of isothermal coordinates is equivalent to the existence of RIemann surface structure on an oriented surface S. In other words, we can think of Korn–Litchenstein theorem as one-dimensional case of **Newlander–Nirenberg theorem**, which states that the integrability of an almost complex structure is equivalent to vanishing of Nijenhuis tensor, which is always the case In dimension 2. For higher dimension, the resulting PDE becomes a over-determined system and thus requires some compatibility conditions, namely vanishing of Nijenhuis tensor.

In the similar vein with dimension 1 case, Frol̈icher in 1956 showed Newlander–Nirenberg theorem using Frobenius theorem (again an ODE method), see Voisin for a details. Later, this was achieved by Newlander–Nirenberg in 1956 using integral kernel method similar here (but of course much more technical). In fact, they required only that the almost complex structure to be C^{2n} and showed that the resulting complex structure is C^{2n+1}. The best regularity result (I know of) is due to Webster in 1989 which asserts that for a $C^{r,\alpha}$ almost complex structure, where $r \geqslant 1$, $\alpha \in (0, 1)$, we can get a $C^{r+1,\alpha}$-complex structure. Later, Joachim Michel (2010) gave a new proof of Webester's result avoiding the use of Nash–Moser iteration in the original proof.

Others equivalent formulations of Theorem 1.1 for smooth metrics are that any surface S is locally conformally flat or u, v are harmonic, i.e., (u, v) is a **harmonic coordinate**. The first statement certainly fails in general for higher dimensions, for some curvature obstructions, see Besse for further discussion. Harmonic coordinates, on the other hand, always exist on any RIemannian manifolds. We refer Taylor's PDE for this approach for proof of Theorem 1.1 with smooth coefficients.

2. Reformulation of the Problem and Preliminaries

Following the idea of Chern, we now reformulate the problem in complex notations. Since ds^2 is positive-definite (i.e., $EG - F^2, E > 0$), we can find 1-form θ_j for $j = 1, 2$ such that $ds^2 = \theta_1^2 + \theta_2^2$. Such θ_1, θ_2 exist since we can take for instance

$$\theta_1 = \sqrt{E}dx + \sqrt{E}Fdy, \quad \theta_2 = \sqrt{\frac{EG - F^2}{E}}dy.$$

If we write $\theta_j = a_j dx + b_j dy$, then any such θ_1, θ_2 satisfy

$$a_1^2 + a_2^2 = E, \quad a_1 b_a + a_2 b_2 = F, \quad b_1^2 + b_2^2 = G.$$

If $\theta_j' = a_j' dx + b_j'$ for $j = 1, 2$ also satisfy $ds^2 = {\theta_1'}^2 + {\theta_2'}^2$, then we have

$$\begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \end{pmatrix} = A \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \text{ or } \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} = A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \text{ for some } A \in \mathrm{O}(2).$$

If we require further that $a_1 b_2 - a_2 b_1 > 0$, then $A \in \mathrm{SO}(2)$. Now, we put $\phi = \theta_1 + i\theta_2$ such that $ds^2 = \phi\overline{\phi}$. From above discussion, we see that such ϕ is unique up to U(1)-action, i.e., multiplication by a complex number of unit modulus. Here comes the important reformulation of the problem.

Lemma 2.1. *The existence of isothermal coordinates (u, v) with respect to ds^2 is equivalent to the existence of complex-valued function $w = u + iv$ such that $\phi = \rho dw$ for some non-zero function ρ.*

Proof. If there exists $w = u + iv$ such that $\phi = \rho dw$, then

$$ds^2 = \phi\overline{\phi} = |\rho|^2 dw d\overline{w} = |\rho|^2(du^2 + dv^2).$$

Conversely, if there exists isothermal coordinates $ds^2 = \lambda(du^2 + dv^2)$, then $\theta_1 = \sqrt{\lambda}du$ and $\theta_2 = \sqrt{\lambda}dv$. Hence, $\phi = \sqrt{\lambda}(du + idv) = \sqrt{\lambda}dw$. □

We now write $z = x + iy$. For a C^1-function, we write $df = \partial_z f dz + \overline{\partial}_z f d\overline{z}$, where

$$dz = dx + idy, \quad d\overline{z} = dx - idy; \quad \partial_z f := \frac{1}{2}\left(\partial_x w - i\partial_y w\right), \quad \overline{\partial}_z f := \frac{1}{2}(\partial_x w + i\partial_y w)$$

we first recall some results related Stokes' theorem and Cauchy integral formula. Fixed $\xi = (s, t) \in \mathbb{C}$, we set $r = |\xi - z|$. For $g \in C^1(\mathbb{R}, \mathbb{R})$, direct computation shows that

$$d\left(g(r)\frac{-(y-t)dy + (x-s)dx}{r^2}\right) = \frac{g'(r)}{r}dx \wedge dy.$$

Let $D \subset \mathbb{R}^2$ be a domain with C^1-boundary $C := \partial D$. By Stokes' theorem, if $\xi \notin D$, we have

$$\int_D \frac{g'(r)}{r}dx \wedge dy = \int_C g(r)\frac{-(y-t)dx + (x-s)dy}{r^2} \tag{2.1}$$

Claim. (2.1) *still holds for $\xi \notin C$ provided LHS converges and $g(0) = 0$.*

Proof of Claim. Indeed, for sufficiently small $\epsilon > 0$, we apply Stokes' theorem to $D \backslash D_\epsilon(\xi)$:

$$\int_{D\backslash D_\epsilon(\xi)} \frac{g'(r)}{r}dx \wedge dy = \int_C g(r)\frac{-(y-t)dx + (x-s)dy}{r^2} - \int_{\partial D_\epsilon(\xi)} g(r)\frac{-(y-t)dx + (x-s)dy}{r^2}$$

Since $r^{-1}g'(r)\mathbf{1}_{D\backslash D_\epsilon(\xi)} \leqslant r^{-1}g'(r)$ on D and $r^{-1}g'(r) \in L^1(D)$ by assumption, Lebesgue dominated convergent theorem shows that

$$\lim_{\epsilon \to 0+} \int_{D\backslash D_\epsilon} \frac{g'(r)}{r}dx \wedge dy = \int_D \frac{g'(r)}{r}dx \wedge dy.$$

On the other hand, we estimate the boundary integral by change of variable: $x = s + \epsilon\cos\theta$, $y = t + \epsilon\sin\theta$ for $\theta \in [0, 2\pi)$:

$$\left|\int_{\partial D_\epsilon(\xi)} g(r)\frac{-(y-t)dx + (x-s)dy}{r^2}\right| \leqslant \int_0^{2\pi} |g(\epsilon)|\frac{\epsilon^2(\cos^2\theta + \sin^2\theta)}{\epsilon^2}d\theta = 2\pi|g(\epsilon)|.$$

Since $g(0) = 0$, $\lim_{\epsilon\to 0} g(\epsilon) = 0$. This establishes the claim. □

Also, we recall that the **winding number** of C about ξ is defined by

$$\nu_C(\xi) = \frac{1}{2\pi i}\int_C \frac{dz}{z - \xi} = \frac{1}{2\pi}\int_C \frac{-(y-t)dx + (x-s)dy}{|z - \xi|^2} \in \mathbb{Z}.$$

If we take $g(r) = r^\alpha$ for $\alpha \neq 0$, then (2.1) becomes

$$\alpha\int_D r^{\alpha-2}dx \wedge dy = \int_C r^{\alpha-2}(-(y-t)dx + (x-s)dy),$$

where the LHS converges if $\alpha > 0$. We present an estimate that will be needed in Lemma 3.1.

Lemma 2.2. *Let $D \subset \mathbb{R}^2$ be a domain with C^1-boundary $C := \partial D$, $\xi = (s,t) \notin C$. Let $k := |\nu_C(\xi)|$. Then*

$$\left|\int_D r^{\alpha-2} dx \wedge dy\right| \leqslant \begin{cases} \frac{2k\pi}{\alpha}\Delta^\alpha, & \alpha > 0 \\ \frac{2k\pi}{-\alpha}\delta^\alpha, & \alpha < 0, \quad \xi \notin D, \end{cases}$$

where $\Delta := \max_{z\in C} |z-\xi|$, $\delta := \min_{z\in C} |z-\xi|$.

Proof. Applying (2.1) to $g(r) = r^\alpha$, we get

$$\left|\int_D r^{\alpha-2} dxdy\right| = \frac{1}{|\alpha|}\left|\int_D r^{-1} g'(r) dxdy\right| \leqslant \frac{1}{|\alpha|}\int_C |g(r)| \left|\frac{-(y-t)dx + (x-s)dy}{r^2}\right| = \frac{1}{|\alpha|}\int_C r^\alpha |d\theta|.$$

The required estimate follows. $\square$

3. Uniform and Hölder Estimate on Cauchy Potential

It is well-known that the "fundamental solution" of $\bar{\partial}_z$ is given by $\Gamma(z) := 1/\pi z$, i.e., $\bar{\partial}_z \Gamma = \delta_0$. Let D be a bounded domain, $f \in L^1(D)$. We define **Cauchy potential** of f by

$$F(\xi) = \Gamma * (\mathbf{1}_D f)(\xi) = \int_D \Gamma(\xi - z) f(z) dx \wedge dy = \frac{1}{\pi}\int_D \frac{f(z)}{\xi - z} dx \wedge dy, \quad \xi \in D. \tag{3.1}$$

A crucial result in Chern's proof is the following uniform and Hölder estimate for Cauchy potential.

Lemma 3.1. *Let $\overline{D} := \{z \in \mathbb{C} : |z| \leqslant R\}$, $f \in C^{0,\alpha}(\overline{D})$ for some $\alpha \in (0,1)$. Then $\partial_\xi F$, $\bar{\partial}_\xi F$ exist and $\bar{\partial}_\xi F = f$. Moreover, if $\|f\|_{L^\infty(D)} < \infty$, then we have*

$$\|F\|_{L^\infty(D)} \leqslant 4R\|f\|_{L^\infty(D)} \tag{3.2}$$

$$\|\partial_\xi F\|_{L^\infty(D)} \leqslant 2^{\alpha+1} R^\alpha [f]_{\alpha,D}/\alpha \tag{3.3}$$

$$|F(\xi_1) - F(\xi_2)| \leqslant 2(\|f\|_{L^\infty(D)} + 2^{\alpha+1} R^\alpha [f]_{\alpha,D}/\alpha)|\xi_1 - \xi_2| \tag{3.4}$$

$$[\partial_\xi F]_{\alpha,D} \leqslant \mu(\alpha)[f]_{\alpha,D}, \tag{3.5}$$

for some constant $\mu(\alpha) > 0$.

Proof. By definition,

$$\begin{aligned}
\bar{\partial}_\xi F &= \lim_{h\in\mathbb{R}, h\to 0} \frac{F(\xi + h/2) - F(\xi) + iF(\xi + ih/2) - iF(\xi)}{h} \\
&= \frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{h}\iint_D \left[\frac{f(z)}{\xi + h/2 - z} - \frac{f(z)}{\xi - z} + i\frac{f(z)}{\xi + ih/2 - z} - i\frac{f(z)}{\xi - z}\right] dz \wedge d\bar{z} \\
&= \frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{2}\iint_D \left[\frac{f(z)}{(\xi - z)(\xi + h/2 - z)} - \frac{f(z)}{(\xi - z)(\xi + ih/2 - z)}\right] dz \wedge d\bar{z} \\
&= \frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{2}\iint_D (f(z) - f(\xi))\left[\frac{1}{(\xi - z)(\xi + h/2 - z)} - \frac{1}{(\xi - z)(\xi + ih/2 - z)}\right] dz \wedge d\bar{z} \\
&\quad + \frac{1}{2\pi i} f(\xi)\frac{\partial}{\partial\bar{\xi}}\iint_D \frac{dz \wedge d\bar{z}}{\xi - z}.
\end{aligned}$$

By assumption that $f \in C^{0,\alpha}(\overline{D})$, we can exchange the limit and the integral and thus the first term is zero. On the other hand,

$$\begin{aligned}
&\iint_D \frac{dz \wedge d\bar{z}}{\xi - z} = 2i\iint_D \frac{dx \wedge dy}{\xi - z} = 2i\int_0^R \int_0^{2\pi} \frac{r d\theta dr}{\xi - z} \\
=& 2i\int_0^R \int_{\partial B_0(r)} \frac{-idz}{z(\xi - z)} dr = 2\int_0^R \int_{\partial B_0(r)} \left(\frac{1}{\xi - z} - \frac{1}{z}\right) dz \cdot \frac{r}{\xi} dr = 2\int_0^{|\xi|} (2\pi i)\frac{r}{\xi} dt = 2\pi i\frac{|\xi|^2}{\xi} = 2\pi i\bar{\xi}.
\end{aligned}$$

Therefore, we conclude that $\bar{\partial}_\xi F = f(\xi)$. By similar a rgument, we obtain

$$\partial_\xi F = \frac{1}{2\pi i}\iint_D \frac{f(z)-f(\xi)}{(\xi-z)^2}dz\wedge d\bar{z}.$$

By Lemma 2.2, we obtain

$$|F| \leqslant \left|\frac{1}{\pi}\iint_D \frac{f(z)}{\xi-z}dx\wedge dy\right| \leqslant \frac{\|f\|_{L^\infty(D)}}{\pi}\iint_D \frac{|z-\xi|}{|z-\xi|^2}dx\wedge dy \leqslant \frac{\|f\|_{L^\infty(D)}(2\pi)(2R)}{\pi} = 4R\|f\|_{L^\infty(D)},$$

since $|z-\xi| \leqslant |z|+|\xi| \leqslant 2R$. Similarly,

$$|\partial_\xi F| \leqslant \frac{[f]_{\alpha,D}}{\pi}\iint_D \frac{|z-\xi|^\alpha}{|z-\xi|^2} \leqslant \frac{[f]_{\alpha,D}}{\pi}\frac{2\pi}{\alpha}(2R)^\alpha = \frac{2^{\alpha+1}}{\alpha}R^\alpha[f]_{\alpha,D}.$$

For Hölder estimate for F, we apply mean-value theorem:

$$|F(\xi_1)-F(\xi_2)| \leqslant |\partial_\xi F||\xi_1-\xi_2| + |\bar{\partial}_\xi F||\bar{\xi}_1-\bar{\xi}_2| \leqslant (2^{\alpha+1}R^\alpha[f]_{\alpha,D}/\alpha + \|f\|_{L^\infty(D)})|\xi_1-\xi_2|.$$

The Hölder estimate for $\partial_\xi F$ is the most complicated one. For $\xi_1,\xi_2 \in B_{R/5}(0)$ so that $B_{2|\xi_1-\xi_2|}(\xi_2) \subset D$, we set $D_0 := D\cap B_{2|\xi_1-\xi_2|}(\xi_2)$. We decompose

$$\begin{aligned}\partial_\xi F(\xi_1)-\partial_\xi F(\xi_2) = &\underbrace{\frac{1}{2\pi i}\iint_{D_0}\frac{f(z)-f(\xi_1)}{(\xi_1-z)^2}dz\wedge d\bar{z}}_{I} - \underbrace{\iint_{D_0}\frac{f((z)-f(\xi_2)}{(\xi_2-z)^2}}_{II}\\ &\underbrace{+\iint_{D\backslash D_0}(f(z)-f(\xi_1))\left(\frac{1}{(z-\xi_1)^2}-\frac{1}{(z-\xi_2)^2}\right)dz\wedge d\bar{z}}_{III}\\ &\underbrace{-f(\xi)-f(\xi_2)\frac{\partial}{\partial x_2}\iint_{D\backslash D_0}\frac{dz\wedge d\bar{z}}{\xi-z}}_{IV}.\end{aligned}$$

By Lemma 2.2 and $|z-\xi_1| \leqslant |z-\xi_2|+|\xi_1-\xi_2| \leqslant 3|\xi_1-\xi_2|$,

$$|I| \leqslant 2[f]_{\alpha,D}\int_{D_0}\frac{|z-\xi_1|^\alpha}{|z-\xi_1|^2} \leqslant \frac{4\pi[f]_{\alpha,D}}{\alpha}(3|\xi_1-\xi_2|)^\alpha.$$

Similarly, $|II| \leqslant \frac{4\pi[f]_{\alpha,D}}{\alpha}(2|\xi_1-\xi_2|)^\alpha$. For III, we have

$$\begin{aligned}|III| =&\left|\iint_{D\backslash D_0}(f(z)-f(\xi_1))\int_{\xi_2}^{\xi_1}\frac{2d\xi}{(z-\xi)^3}2idxdy\right|\\ \leqslant&4\int_{\xi_2}^{\xi_1}\left(\left|\iint_{D\backslash D_0}\frac{f(z)-f(\xi)}{(z-\xi)^3}dxdy\right| + \left|\iint_{D\backslash D_0}\frac{f(\xi_1)-f(\xi)}{(z-\xi)^3}dxdy\right|\right)d|\xi|\\ =&4\int_{\xi_2}^{\xi_1}\left([f]_{\alpha,D}\iint_{D\backslash D_0}\frac{|z-\xi|^{\alpha-1}}{|z-\xi|^2}dxdy + [f]_{\alpha,D}\iint_{D\backslash D_0}\frac{|\xi_1-\xi|^\alpha}{|z-\xi|^3}dxdy\right)d|\xi|\\ \leqslant&4\int_{\xi_2}^{\xi_1}\left(\frac{[f]_{\alpha,D}2\pi\cdot 2}{1-\alpha}|\xi_1-\xi_2|^{\alpha-1} + [f]_{\alpha,D}|\xi_1-\xi_2|^\alpha + \frac{2\pi\cdot 2}{1}|\xi_1-\xi_2|^{-1}\right)|d\xi|,\end{aligned}$$

for $|z-\xi| \geqslant |z-\xi_2|-|\xi-\xi_2| \geqslant 2|\xi_1-\xi_2|-|\xi_1-\xi_2| = |\xi_1-\xi_2|$. Hence, we get

$$|III| \leqslant 16\pi\left(\frac{1}{1-\alpha}+1\right)[f]_{\alpha,D}|\xi_1-\xi_2|^\alpha$$

Finally, for IV, we notice that

$$\iint_{D\backslash D_0}\frac{dz\wedge d\bar{z}}{\xi_2-z} = \iint_D \frac{dz-\wedge d\bar{z}}{z-\xi_2} - \iint_{D_0}\frac{dz\wedge d\bar{z}}{z-\xi} = 2\pi\bar{\xi}_2 + 0.$$

Hence, $IV = 0$. We may set $\mu(\alpha)$ to be

$$\mu(\alpha) := \frac{2}{\alpha}3^\alpha + \frac{2}{\alpha}2^\alpha + 8\left(\frac{1}{1-\alpha} + 1\right).$$

□

4. Proof of Main Theorem

Chern's approach reduces the problem of isothermal coordinates to existence and uniqueness of certain integral equations. More preciesely, he proved:

Theorem 4.1 (Chern)**.** *Let $\overline{D} := \{z \in \mathbb{C} : |z| \leqslant R\}$ be a closed disk. Let $Zf = a\bar{\partial}_z f + b\partial_z f + cf$ be a linear operator of first order, where $a, b, c \in C^{0,\alpha}(D)$ and $a(0) = b(0) = 0$. Given a holomorphic function σ on $\overline{D}$ with $\sigma(0) = 0$, for sufficiently small $R \ll 1$, the inhomogenous integral equation*

$$w(\xi) + \frac{1}{2\pi i}\int_D \frac{Zw}{z-\xi} dz \wedge d\bar{z} = \sigma(\xi), \quad \xi \in D, \tag{4.1}$$

has a unique solution w with $Zw \in C^{0,\alpha}(\overline{D})$.

We now deduce Theorem 1.1 from Theorem 4.1.

Proof of Theorem 1.1 (assuming Theorem 4.1). Since the problem is local, we may WLOG assume $0 \in D$ and we prove the statement for $p = 0$. Up to a linear change of coordinate, we may assume $E(0) = G(0) = 1$, $F(0) = 0$. Thus, we get

$$(\theta_1)_0 = dx, \quad (\theta_2)_0 = dy, \quad (\phi)_0 = dx + idy = dz.$$

In other words, complex-valued 1-form ϕ is given by

$$\phi = (1 - a(z))dz + b(z)d\bar{z}, \quad a(0) = b(0) = 0.$$

By Lemma 2.1, existence of isothermal coordinates (u, v) is equivalent to solution of $w = u + iv$ to $\phi = \rho dw$. By above discussion, this equation is equivalent to $\bar{\partial}_z w = b/\rho$ and $\partial_z w = (1-a)/\rho$, or equivalently, $Zw := a\bar{\partial}_z w + b\partial_z w = \bar{\partial}_z w$.

Since $a, b \in C^{0,\alpha}(D)$ with $a(0) = b(0) = 0$, for any holomorphic function σ on D with $\sigma(0) = 0$, there exists a solution $w \in C^{0,\alpha}(D)$ to the integral equation by Theorem 4.1

$$w(\xi) + \frac{1}{2\pi i}\int_D \frac{Zw}{z-\xi} dz \wedge d\bar{z} = \sigma(\xi).$$

Since $Zw \in C^{0,\alpha}(D)$, $\bar{\partial} w = Zw$ by Lemma 3.1. This proves Theorem 1.1. □

Let us wrap up the notation and give some preliminary estimate. Let $\overline{D} := \{|z| \leqslant R\}$. Given $a, b \in C^{0,\alpha(\overline{D})}$ for $\alpha \in (0,1)$ and $\sigma \in \mathcal{O}(D)$ with $a(0) = b(0) = \sigma(0) = 0$. We consider the following first order differential operator

$$Z = a\frac{\partial}{\partial \bar{z}} + b\frac{\partial}{\partial z}.$$

Notice that $f \in C^{0,\alpha}(\overline{D})$ and $g \in C^1(D)$, $fg \in C^{0,\alpha}(D')$ for possibly smaller $D' \Subset D$. Hence, by shrinking D if necessary, we may assume

$$(Z + c)\sigma = a\sigma + b\sigma' \in C^{0,\alpha}(D).$$

For $M > 1$ such that $\|c\|_{L^\infty(D)} \leqslant M$, $\|\sigma\|_{L^\infty(D)} \leqslant M$, we have for $\xi_1, \xi_2 \in D$,

$$\begin{aligned} &|h(\xi_1) - h(\xi_2)| \leqslant M|\xi_1 - \xi_2|^\alpha, \quad h = a, b, c, \sigma \\ &|(Z+c)\sigma(\xi_1) - (Z+c)\sigma(\xi_2)| \leqslant \frac{M^2}{2^\alpha}|\xi_1 - \xi_2|^\alpha. \end{aligned} \tag{4.2}$$

Hence, $a(0) = b(0) = (Z + c)\sigma(0) = 0$ implies that

$$|a(\xi)|, |b(\xi)| \leqslant M|\xi|^{\alpha} \leqslant MR^{\alpha}; \quad |(Z+c)\sigma| \leqslant \frac{M^2}{2^{\alpha}} R^{\alpha} \leqslant M^2 R^{\alpha}. \tag{4.3}$$

We also need the following a priori estimate for Cauchy potential.

Lemma 4.1. *For $f \in C^{0,\alpha}(\overline{D})$, $[f]_{\alpha,D} \leqslant K$, the Cauchy potential*

$$F(\xi) = \frac{1}{2\pi i} \iint_D \frac{f(z)}{\xi - z} dz \wedge d\bar{z}$$

satisfies the following uniform and Hölder estimate

$$\|(Z+c)F\|_{L^\infty(D)} \leqslant MR^{\alpha}((1 + 4R^{1-\alpha})\|f\|_{L^\infty(D)} + \frac{2^{\alpha+1}}{\alpha} R^{\alpha}[f]_{\alpha,D}) \tag{4.4}$$

$$|(Z+c)F(\xi_1) - (Z+c)F(\xi_2)| \leqslant M(\|f\|_{L^\infty(D)} + \|f\|_{L^\infty(D)}(g_1(R) + [f]_{\alpha,D} g_2(R)))|\xi_1 - \xi_2|^{\alpha}, \tag{4.5}$$

for some $g_1(R), g_2(R)$ depends only on α and R.

Proof. By Lemma 3.1, we get

$$\begin{aligned}|(Z+c)F| = |a\bar{\partial}_{\xi}F + b\partial_{\xi}F + cF| \leqslant& MR^{\alpha}\|f\|_{L^\infty(D)} + MR^{\alpha}\frac{2^{\alpha+1}}{\alpha}R^{\alpha}[f]_{\alpha,D} + 4MR\|f\|_{L^\infty(D)} \\ =& MR^{\alpha}((1+4R^{1-\alpha})\|f\|_{L^\infty(D)} + 2^{\alpha+1}R^{\alpha}[f]_{\alpha,D}/\alpha).\end{aligned}$$

This proves (4.4). For (4.5),

$$\begin{aligned}
&|(Z+c)F(\xi_1) - (Z+c)F(\xi_2)| \\
=&|af(\xi) - af(\xi_2) + b\partial_{\xi}F(\xi_1) - b\partial_{\xi}F(\xi_2) + cF(\xi_1) - cF(\xi_2)| \\
\leqslant&|(a(\xi) - a(\xi_2))f(\xi_1)| + |a(\xi_2)(f(\xi_1) - f(\xi_2))| \\
&+|(b(\xi_1) - b(\xi_2))\partial_{\xi}F(\xi_1)| + |b(\xi_2)(\partial_{\xi}F(\xi_1) - \partial_{\xi}F(\xi_2))| \\
&+|(c(\xi_1) - c(\xi_2))F(\xi_1)| + |c(\xi_2)(F(\xi_1) - F(\xi_2))| \\
\leqslant&M\|f\|_{L^\infty}|\xi_1 - \xi_2|^{\alpha} + MR^{\alpha}[f]_{\alpha,D}|\xi_1 - \xi_2|^{\alpha} + M|\xi_1 - \xi_2|^{\alpha}\frac{2^{\alpha+1}}{\alpha}R^{\alpha}[f]_{\alpha,D} + MR^{\alpha}\mu(\alpha)[f]_{\alpha,D}|\xi_1 - \xi_2|^{\alpha} \\
&+M|\xi_1 - \xi_2|^{\alpha}4R\|f\|_{L^\infty} + 2M(\|f\|_{L^\infty} + 2^{\alpha+1}R^{\alpha}[f]_{\alpha,D}/\alpha)|\xi_1 - \xi_2|^{\alpha} \\
=&M|\xi_1 - \xi_2|^{\alpha}(\|f\|_{L^\infty} + \|f\|_{L^\infty}\underbrace{4R + 2^{2-\alpha}R^{1-\alpha}}_{g_1(R)} + [f]_{\alpha,D}(\underbrace{1 + 2^{\alpha+1}/\alpha + \mu(\alpha)R^{\alpha} + 8R/\alpha}_{g_2(R)})).
\end{aligned}$$

□

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We construct the solution by recursion:

$$w_0(\xi) = \frac{1}{2\pi i}\sigma(\xi), \quad w_{n+1}(\xi) = \frac{1}{2\pi i}\iint_D \frac{(Z+c)w_n(z)}{z - \xi} dz \wedge d\bar{z}.$$

We first establish the following estimates:

Claim. *There exists $C = C(\alpha) > 1$ such that*

$$\|w_n\|_{L^\infty(D)} \leqslant M(CMR^{\alpha})^n, \quad \|(Z+c)w_n\|_{L^\infty(D)} \leqslant M(CMR^{\alpha})^{n+1} \tag{4.6}$$

$$[w_n]_{\alpha,D} \leqslant M(cMR^{\alpha})^n, \quad [(Z+c)w_n]_{\alpha,D} \leqslant \frac{CM^2}{2^{\alpha}}(CMR^{\alpha})^n. \tag{4.7}$$

Proof of Claim. For $n = 0$, this is already established in (4.3). Now, by Lemma 3.1 and induction hypothesis, $|w_{n+1}| \leqslant 4RM(CMR^\alpha)^{n+1} \leqslant M(CMR^\alpha)^{n+1}$ if we choose $4R \leqslant 1$. Also,

$$\begin{aligned}
|w_{n+1}(\xi_1) - w_{n+1}(\xi_2)| &\leqslant 2\left(M(CMR^\alpha)^{n+1} + \frac{2^{\alpha+1}}{\alpha}R^\alpha \cdot \frac{CM^2}{2^\alpha}(CMR^\alpha)^n\right)|\xi_1 - \xi_2| \\
&\leqslant M(CMR^\alpha)^{n+1|}|\xi_1 - \xi_2|^\alpha(2(1+2/\alpha)|\xi_1 - \xi_2|^{1-\alpha}) \\
&\leqslant M(CMR^\alpha)^{n+1}|\xi_1 - \xi_2|^\alpha \cdot 2 \cdot \frac{\alpha+2}{\alpha}(2R)^{1-\alpha} \\
&\leqslant M(CMR^\alpha)^{n+1}|\xi_1 - \xi_2|^\alpha.
\end{aligned}$$

By (4.4), we get

$$\begin{aligned}
|(Z+c)w_{n+1}| \leqslant & MR^\alpha((1+4R^{1-\alpha})\|w_{n+1}\|_{L^\infty}) + \frac{2^{\alpha+1}}{\alpha}R^\alpha[w_{n+1}]_{\alpha,D} \\
\leqslant & MR^\alpha((1+4R^{1-\alpha})M(CMR^\alpha)^{n+1} + \frac{2^{\alpha+1}}{\alpha}R^\alpha \cdot \frac{CM^2}{2^\alpha}(CMR^\alpha)^n) \\
= & M(CMR^\alpha)^{n+2} \cdot C^{-1}(1+4R^{1-\alpha} + 2/\alpha) \\
\leqslant & M(CMR^\alpha)^{n+2}C^{-1}((\alpha+2)/\alpha + 2^\alpha\alpha/(\alpha+2)) \leqslant M(CMR^\alpha)^{n+2}.
\end{aligned}$$

Finally, by (4.5),

$$\begin{aligned}
&|(Z+c)w_{n+1}(\xi_1) - (Z+c)w_{n+1}(\xi_2)| \leqslant M(\|w_n\|_{L^\infty} + \|w_n\|(g_1(R)) + [w_n]_{\alpha,D}g_2(R))|\xi_1 - \xi_2|^\alpha \\
\leqslant & M|\xi_1 - \xi_2|^\alpha\left(M(CMR^\alpha)^{n+1}(1+g_1(R)) + CM^2 2^{-\alpha}(CMR^\alpha)^n g_2(R)\right) \\
= & \frac{CM^2}{2^\alpha}(CMR^\alpha)^{n+1}|\xi_1 - \xi_2^\alpha C^{-1}(2^\alpha(1+g_1(R)) + R^{-\alpha}g_2(R)).
\end{aligned}$$

If one plug in $g_1(R), g_2(R)$ as in (4.5), then one can show that

$$2^\alpha(1+g_1(R)) + R^{-\alpha}g_2(R) \leqslant 1 + \mu(\alpha) + \frac{3\alpha+2}{\alpha}2^\alpha \leqslant C.$$

□

By (4.7), $(Z+c)w_n \in C^{0,\alpha}(D)$ and thus w_{n+1} is defined. By (4.6), $w = \sum_{n=0}^\infty w_n$ converges absolutely and uniformly provided $R \ll 1$ such that $CMR^\alpha < 1$. Hence, we can commute summation with integral sign and differentiation:

$$2\pi i w(\xi) - \iint_D \frac{(Z+c)w(z)}{z-\xi}dz \wedge d\bar{z} = \sigma(\xi).$$

Now, for uniqueness, if $\widetilde{w}$ is another solution of the integral equation (4.1) such that $(Z+c)\widetilde{w} \in C^{0,\alpha}(D)$, then $\widehat{w} := w - \widetilde{w}$ satisfies

$$2\pi i\widehat{w}(\xi) = \iint_D \frac{(Z+c)\widehat{w}(z)}{z-\xi}dz \wedge d\bar{z}.$$

□