A SURVEY ON CLASSICAL MIRROR SYMMETRY

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Abstract. We survey the ingredients and results in classical mirror symmetry. In particular, we present the Hodge-theoretic version of mirror symmetry and its relation to others versions. In the end, we state mirror theorem on quintic 3-fold. This is a part of report of research assistant program under supervision of professor Chin-Lung Wang, funded by Ministry of Science and Technology program: 107-2115-M-002-002-MY3.

1. Introduction

Mirror symmetry arises when physicists studied string theory in the late 1980s. They discovered two kinds of physical theories (superconformal field theories), called A-model and B-model respectively, on different Calabi-Yau 3-folds X and $\check{X}$ are identical under certain transformation. This pair of Calabi-Yau 3-fold X and $\check{X}$ are said to be a *mirror pair*. In 1990, Greene and Plesser [9] predicted the simplest mirror pair of Calabi-Yau 3-folds, the quintic 3-folds. We now explain what these discoveries in physics in mathematical context.

To our later purpose, we first begin with a definition of singular Calabi-Yau variety.

Definition 1. An n-dimensional Calabi-Yau variety is a projective normal variety X such that

(1) X has at worst Gorenstein canonical singularities
(2) The dualizing sheaf is trivial, i.e. $\omega_X \cong \mathcal{O}_X$
(3) $H^i(X, \mathcal{O}_X) = 0$, for $i = 1, \dots, n-1$.

In particular, when X is smooth, we require it to be simply-connected. On a Calabi-Yau 3-fold X, we have a complexified Kähler form $\omega^{\mathbb{C}} = B + \sqrt{-1}\omega$, which is a closed $(1,1)$-form and the imaginery part is non-degenerate. Also, since X ia a complex manifold, we have a complex structure J on X. Then A-model is physical theory on the moduli space of complexfied Kähler form, and B-model is on the moduli space of complex structure. Since the deformation of Kähler form is unobstructed, it is governed by $H^{1,1}(X)$. On the other hand, by Bogormolov-Tian-Todorov theorem, the deformation of complex structure on Calabi-Yau manifolds is also unobstructed and it is governed by $H^{2,1}(X)$. Therefore, the first mathematical consequence of mirror symmetry is the following

$$H^{2,1}(\check{X}) = H^{1,1}(X)$$
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Date: October 21, 2019.

Moreover, the assertions that two physical theories are identical implies that their three-point correlation functions on $H^{1,1}$ and on $H^{2,1}$ are coincide under some transformation. In A-model, the three-point correlation function on (X, ω) is given by

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{\beta \in H^2(X,\mathbb{Z})\setminus 0} n_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \frac{\exp(2\pi\sqrt{-1}\int_\beta \omega)}{1 - \exp(2\pi\sqrt{-1}\int_\beta \omega)},$$

where $\omega_i \in H^2(X, \mathbb{C})$, and n_β is the "number of rational curves in X representing the homology class β." Note that the A-model correlation function is the quantum correction of cup product on X. The notion of n_β has been rigorously defined by Gromov-Witten theory and qunatum cohomology. We refer construction of Gromov-Witten invariants to [3] and the references therein.

On the other hand, in B-model on $\check{X}$ and a holomorphic volume form Ω on it, the three point correlation function, called Yukawa coupling is given by

$$\langle \theta_1, \theta_2, \theta_3 \rangle = \int_{\check{X}} \Omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega,$$

where ∇ is the Gauss-Manin connection on the complex moduli of $\check{X}$. Thus, mirror symmetry can be formulated as: The A-model correlation function on X and the Yukawa coupling on $\check{X}$ are identical under suitable coordinate transformation, called *mirror map*. For the case of quintic 3-fold, Candelas et al.[2] calculated the number n_β above from B-model via mirror symmetry. Later, in 1996-1997, Givental [8] and Liu-Lian-Yau [12] proved the case of quintic 3-fold rigorously. However, the story is more complicated. Actually, the mirror symmetry only holds locally near a certain boundary point of complexified Kähler moduli space of X and complex moduli space of $\check{X}$.

On the other hand, one can actually formulate a mathematical statement of mirror symmetry, which is equivalent to the original version of mirror symmetry in terms of correlation functions. This is called *Hodge-theoretic mirror symmetry* defined in sec.4.2, which asserts that mirror symmetry is an equivalence of two variation of Hodge structure. In B-model, this is just variation of Hodge structure arising from geometry. However, in A-model, we need to establish a variation of Hodge structure.

Now, we give structure of the article briefly. In section 2, we collect some facts on Hodge theory for our later use. We also define the large complex structure limit point, which is the right kind of boundary point for the mirror symmetry to hold. Later, we show that there exists canonical coordinate on complex moduli and put Yukawa coupling in a more canonical form. Next, in section 3, we turn to Gromov-Witten invariants and quantum cohomology on Calabi-Yau 3-folds. In particular, We establish the variation of Hodge structure for A-model. Later, in section 4, we define the mirror map and give the precise statement for Hodge-theoretic mirror symmetry. We also give Batyrev's toric recipe to construct mirror manifold. Finally, in section 5, we specialize to quintic 3-folds and its mirror, and state the precise statement of mirror theorem on quintic in [12].

2. B-Model: Variations and Degenerations of Hodge Structures

2.1. **Variations and Degenerations of Hodge Structures.** Given a Calabi-Yau 3-fold $\check{X}$, we denote $\mathcal{M}(\check{X})$ to be the complex moduli space of $\check{X}$. By Bogormolov–Tian–Todorov unobstructed theorem, $\mathcal{M}(\check{X})$ is smooth and has dimension $h^{2,1}(\check{X})$.

Now, we consider product of sufficiently small punctured disks $S := (\Delta^*)^r \subset \mathcal{M}(\check{X})$, and the family $\pi : \mathcal{X} \to S$, which each fiber X_s are all Calabi-Yau 3-folds.

Remark 1. The following is an explanation that why the sufficiently small deformation of Calabi-Yau manifolds are still Calabi-Yau. It is well-known that the sufficiently small deformation of compact Kähler manifolds are Kähler.([18], Thm 9.23)

Now, if we are given a family $\pi : \mathcal{X} \to S$ with a distinguished fiber X_0 having trivial canonical bundle. One can use the degeneracy of Hodge-de Rham spectral sequence at E_1 for the neighboring fiber to show that the Hodge numbers are locally constant functions on S. So, for s near 0, $H^{p,q}(X_s) = H^{p,q}(X_0) \cong \mathbb{C}$. Since our definition of Calabi-Yau manifold includes $H^1(X, \mathcal{O}_X) = 0$, and X_s is diffeomorphic to X_0. We know that the canonical bundle on X_s is topologically trivial, and the long exact sequence from the exponential sequence shows that the canonical bundle on X_s is also holomorphically trivial. If one drop the condition of $H^1(X, \mathcal{O}_X) = 0$, then Tsung-Ju pointed out to me that this is still true. However, it requires a result from Deligne.

Next, we consider the variation Hodge structure (VHS) of weight 3 over S of $H^3(\check{X}, \mathbb{C})$. It consists of a holomorphic bundle $\mathcal{H}_{\check{X}}$ assocaited to the local system $R^3\pi_*\mathbb{C}$ with real structure $\mathcal{H}_{\mathbb{R}}$ coming from the local system $R^3\pi_*\mathbb{R}$, a flat connection ∇^{GM}, called *Gauss-Manin connection*, and a finite decreasing filtration $\mathcal{F}^\bullet$ of holomorphic sub-bundle of $\mathcal{H}_{\check{X}}$, called *Hodge filtration*, satisfying

(1) $\nabla^{GM}\mathcal{F}^p \subset \Omega^1_S \otimes \mathcal{F}^{p-1}$ (Griffths' transversality) and
(2) $\mathcal{H} = \mathcal{F}^p \bigoplus \overline{\mathcal{F}^{3-p+1}}$.

We can also decompose $\mathcal{H}_{\check{X}}$ as C^∞-bundle by

$$\mathcal{H}_{\check{X}} = \mathcal{H}^{p,q},$$

where $\mathcal{H}^{p,q} = \mathcal{F}^p \cap \overline{\mathcal{F}^q}$, where the conjugate is induced from the real structure.

The *polarization* $\mathcal{Q}$ is a flat non-degenerate skew-symmetric bilinear form on $\mathcal{H}_{\check{X}}$ and is given by

$$Q(\alpha, \beta) = -\int_{X_s} \alpha \cup \beta,$$

on each fiber. Then Hodge-Riemann bilinear relations asserts induced hermitian metric $\mathcal{Q}^h := (-i)^3\mathcal{Q}(\cdot, \bar{\cdot})$ makes the decomposition orthogonal and such that $(-1)^3\mathcal{Q}^j$ is positive definite on $\mathcal{H}^{p,3-p}$.

Alternatively, via parallel transporting of a fixed fiber $V := H^3(X_s, \mathbb{C})$, we can describe a variation of polarized Hodge structure by *period map* $\phi : S \to D/\Gamma$, where D is the classifying space of polarized Hodge structure of weight 3, and Γ is the monodromy group. Recall that D is the Zariski open set of the smooth projective variety $\check{D}$ that parametrizing all filtration $F^\bullet$ in V with fixed Hodge number and satisfying first Hodge-Riemann bilinear relation. We also have $G_{\mathbb{C}} := \mathrm{Aut}(V, Q)$ acting transitively on $\check{D}$ and D is an open orbit of $G_{\mathbb{R}} := \mathrm{Aut}(V_{\mathbb{R}}, Q)$.

We denote $\bar{S} := \Delta^r$ to be the compactification of S and $\tilde{S} := U^r$ to be the universal covering of S, which is the upper-half space. Suppose $t = (t_i), z = (z_j)$ are coordinates on $U^r, (\Delta^*)^r$ resepctively, and we have $z_j = \exp(2\pi\sqrt{-1}t_j)$. Then the period map lifts to a map $\Phi : U^r \to D$. $D := \bar{S} - S$ is a normal crossing divisor defined by $z_1 \dots z_r = 0 = \cup_{i=1}^r D_i$, where $D_i = (z_i = 0)$. Let $\mathcal{T}_j$ be the monodromy operator given by parallel transporting on a loop around D_j, then $\mathcal{T}_j \in G_{\mathbb{R}}$ and the lifting path acts on Φ by:

$$\Phi(t + e_j) = \mathcal{T}_j \Phi(t)$$

The monodromy theorem ([16], 6.1) asserts that T_i are quasi-unipotent with unipotency index at most $\dim \check{X} + 1 = 4$. By passing to covering of S, we may assume that $\mathcal{T}_i$ are all unipotent operators. Therefore, we can define the monodromy logarithms

$$N_j = \log(T_j) := \sum_{m=1} (-1)^{m+1} \frac{(\mathcal{T}_j - I)^m}{m!}.$$

Since $\exp(\sum_j t_j N_j)$ lies in $G_{\mathbb{C}}$, the map

$$\widetilde{\Psi}(t) = \exp(-\sum_j t_j N_j) \cdot \Phi(t)$$

is a $\check{D}$-valued holomorphic map. Also, one can see directly that $\widetilde{\Psi}(t + e_j) = \widetilde{\Psi}(t)$, it descends to a holomorphic map $\Psi : (\Delta^*)^r \to \check{D}$ such that $\widetilde{\Psi}(t) = \Psi(e^{2\pi\sqrt{-1}t})$.

Then Schmid's nilpotent orbit theorem ([16]) asserts that Ψ extends holomorphically to $\bar{S}$. The image at 0 $F_{\lim} := \Psi(0) \in \check{D}$ is called *the limiting Hodge filtration.* Moreover, the map $\theta : U^r \to \check{D}$ given by

$$\theta(t) := \exp\left(\sum_{j=1}^r t_j N_j\right) \cdot F_{\lim} \in \check{D} \tag{1}$$

defines a period map for $\mathrm{Im}(t_j)$ sufficiently large. (In particular, θ is D-valued for $\mathrm{Im}(t_j) \gg 0$.) We call θ the *nilpotent orbit.*

For any nilpotent operator N on $\mathfrak{gl}(V_{\mathbb{R}})$ defines a increasing filtration $W_\bullet(N)$ which is unique characterized by

(1) $N(W_k) \subset W_{k-2}$,
(2) N^k induces isomorphism between $Gr_{3+k}(W_\bullet) \to Gr_{3-k}(W_\bullet)$,

It is easy to see that that (1) and (2) of the above properties determines $W_\bullet$ uniquely. First, from (2), $N^3 : W_6/W_5 \xrightarrow{\cong} W_0$. Hence, $W_5 = \ker(N^3)$ and $W_0 = \mathrm{im}(N^3)$ Then replace $W_6 = V$ by W_5/W_0, and the induced operator N on it. Now, it is nilpotent of index 3. Since N^2 induces isomorphism between W_5/W_4 and W_1/W_0, we see that W_1 is the preimage of $\mathrm{im}(N^2)$ in W_5/W_0, and W_4 is the preimage of $\ker(N^2)$ in W_5/W_0. Inductively, we can determine all W_k uniquely.

Also, following a theorem of Cattani and Kaplan [3] that if we choose N from the simplicial cone $\mathcal{N} \subset \mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}})$ spanned by the monodromy logarithms $N_1, \dots, N_r$, then the induced filtration is the same. In this case, we just denote $W_\bullet$ and call it *monodromy weight filtration.*

Here, we collect some terminologies and facts on mixed Hodge structure. First, every mixed Hodge structure (W,F) on V has a canonical bigrading $\{I^{p,q}\}$ of V, which is unique characterized by $I^{p,q} \equiv \overline{I^{q,p}} \mod (\bigoplus_{a<p,b<q} I^{a,b})$, such that $W_k = \bigoplus_{p+q\leq k} I^{p,q}$ and $F^r = \bigoplus_q \bigoplus_{p\geq r} I^{p,q}$. A MHS is splits over $\mathbb{R}$ if $I^{p,q} = \overline{I^{q,p}}$. In this case, the bigrading is given by $I^{p,q} = F^p \cap \overline{F^q} \cap W_{p+q}$ Also, it is said to be *Hodge-Tate* type if $I^{p,q} = 0$ if $p \neq q$. Finally, given V, $V_\mathbb{R}$ and a non-degenerate $\mathbb{R}$-bilinear form Q with parity $(-1)^k$, we say a mixed Hodge structure (W,F) is *polarized* by $N \in \mathfrak{g}_\mathbb{R}$ (i.e., N is an infinitesimal automorphism of Q) if

(1) N has nilpotency index $k+1$,
(2) W is a monodromy weight filtration for (N,k),
(3) F satisfies first Hodge-Riemann bilinear relation, i.e.,$Q(F^p, F^{k-p+1}) = 0$,
(4) $N(F^p) \subset F^{p-1}$.
(5) The Hodge structure of weight $k+l$ induced by F on $\ker(N^{l+1} : Gr^W_{k+l} \to Gr^W_{k-l-2}$ is polarized by $S(\cdot, N^k\cdot)$.

Then Schmid ([16], 6.16) proved that the monodromy weight filtration $W_\bullet$ and limiting Hodge structure $F^\bullet_{\lim}$ forms a mixed Hodge structure, and it is polarized by any $N \in \mathcal{N}$. Conversely, if we are given commuting nilpotent operator $N_1, \dots N_r \in \mathfrak{g}_\mathbb{R}$ such that (W,F) is polarized by any N in the cone spanned by $N_1, \dots, N_r$, then $\exp(\sum_i t_i N_i) \cdot F$ is a nilpotent orbit.

We end this section by giving an example of polarized mixed Hodge structure. This example is central in constructing variation of Hodge structure of A-model.

Example 1. Let M be a compact Kähler manifold of dimension n and $V = H^*(M,\mathbb{C})$. Then the bigrading $I^{p,q} = H^{n-p,n-q}(M)$ defines a mixed Hodge structure on V that splits over $\mathbb{R}$. Then $W_k = \bigoplus_{p+q\leq k} H^{p,q}(M)$ and $F^p = \bigoplus_a \bigoplus_{b\geq p} H^{a,b}(M)$. Hard Lefschetz theorem translates directly that for any Kähler class ω on M, W is the monodromy weight filtration for the Lefschetz operator $L_\omega := \omega\cup$. Also, Hodge-Riemann bilinear relation gives the polarization of (F,W) by L_ω with respect to the intersection form. When we restrict (W,F) to $\bigoplus_p H^{p,p}(M)$, they define a mixed Hodge structure of Hodge-Tate type.

2.2. **Large Complex Structure Limit Point.** As we have mentioned in the introduction, mirror symmetry is a statement about local correspondence of mirror pair of Calabi-Yau 3-folds near the boundary point of complex moduli space and Kähler moduli space respectively. Hence, we need to specify what kinds of boundary points on the complex moduli we are interested. As in sec.2.1, $\check{X}$ is a Calabi-Yau 3-folds, $(\Delta^*)^r \subset \mathcal{M}(\check{X})$, and Δ^r is a compactification of $(\Delta^*)^r$.

Definition 2. $0 \in \Delta^r$ is called the *large complex structure limit point* (LCSL point for short) in the complex moduli of $\check{X}$ if

(1) all monodromy operators are unipotents,
(2) for any $N \in \mathcal{N}$, let $W_\bullet$ be the associated monodromy weight filtration, then $\dim W_0 = \dim W_1 = 1$ and $\dim W_2 = r$.
(3) take $g_0, \dots, g_r$ be a basis of W_2 with g_0 generates W_0. For each j,k, define $m := (m_{jk})$ by: $N_j(g_k) = m_{jk} g_0$. Then m is invertible.

Remark 2. The condition $\dim W_0 = 1$ and $N^3 : W_6/W_5 \xrightarrow{\cong} W_0$ imply that $N^3 \neq 0$. By monodromy theorem, we must have $N^4 = 0$. Hence, the monodromy operator N has the maximal unipotentcy index. Thus, in some literatures, for instance [5] and [14], the LCSL point is also called *maximal unipotent boundary point.*

Here are some implications and explanations for the definition of large complex structure limit point. First of all, from $Gr_{3+k}(W_\bullet) \cong Gr_{3-k}(W_\bullet)$, we can see that $\dim W_3 = r+1, \dim W_4 = 2r+1$, and $\dim W_5 = 2r+1$. Since the Hodge filtration $\mathcal{F}^\bullet$ extends holomorphically to 0, the dimensions of graded quotients of $F^\bullet_{\text{lim}}$ are still $1, r, r, 1$. Also, $(F^\bullet_{\text{lim}}, W_\bullet)$ forms a mixed Hodge structure. Combining these two, we find that $Gr_{2k}(W_\bullet)$ has only (k,k)-part. Thus, the mixed Hodge diamond for $(F^\bullet_{\text{lim}}, W_\bullet)$ is:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & r & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & r & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Hence, at LCSL point, the mixed Hodge structure formed by the monodromy weight filtration and limiting Hodge structure is Hodge-Tate.

For any $g_0 \in W_0$, we denote $\gamma_0 \in H_3(X, \mathbb{C})$ be the dual of g_0 and $y_0 = \langle g_0, \Omega \rangle := \int_{\gamma_0} \Omega$. Since $W_0 = \operatorname{im}(N^3)$, for any N_j, $N_j(g_0) = 0$. Therefore, for any monodromy operator T_j, $T_j(g_0) = g_0$, i.e. g_0 is invariant under monodromy. Therefore, the period y_0 is invariant under monodromy, and hence extends over boundary holomorphically. We see that the condition $\dim W_0 = 1$ implies that at large complex structure limit point, Picard-Fuchs equations has unique holomorphic solution. Moreover, since ∇^{GM} has regular singularities along boundary divisor, other $g_k \in W_2$ gives a solution $y_k = \langle g_k, \Omega \rangle$ of the Picard-Fuchs equation with at worst logarithmic growth along $D_j's$. Hence, $\dim W_2 = r+1$ means we have r linearly independent solution with logarithmic pole along divisors.

2.3. **Yukawa Coupling.** In the physicist's version of mirror symmetry, the correlation function on B-model is called *Yukawa coupling* on the complex moduli $\mathcal{M}(\check{X})$ of Calabi-Yau 3-fold $\check{X}$ with a chosen nowhere-vanishing holomorphic n-form Ω, which is defined by:

$$\langle \delta_i, \delta_j, \delta_k \rangle := \int_{\check{X}} \Omega \wedge \nabla_{\delta_i} \nabla_{\delta_j} \nabla_{\delta_k} \Omega,$$

where $\delta_i := z_i \frac{\partial}{\partial z_i}$ is the logarithmic derivative. Since we are working over near the large complex structure limit 0, ∇ has regular singular point at 0. We use logarithmic derivative instead.

Notice that the Yukawa coupling is not unique. It depends on the choice of the holomorphic 3-form Ω and moduli coordinate. In mirror symmetry, to match up the correlation functions between Kähler moduli space of X and complex moduli space of $\check{X}$, we need to specify a *canonical coordinate* on $S = (\Delta^*)^r$ and a particular normalization for the holomorphic 3-form Ω. In [5], they only do the 1-dimensional family in full detail. Here, we deal with the multi-dimensional case directly.

First, we normalize the holomorphic 3-form Ω as following. Since we are dealing in the neighborhood of large complex structure limit of $\mathcal{M}(\check{X})$, the first term of monodromy weight filtration W_0 is one dimensional. Let g_0 be a generator of $W_0 \cap H^3(\check{X}, \mathbb{Z})$, which is unique up to a sign, and γ_0 is the Poincaré dual of g_0. Then we have seen that the period $y_0 = \langle g_0, \Omega \rangle = \int_{\gamma_0} \Omega$ is the unique holomorphic solution of Picard-Fuchs equations at boundary point $0 \in \Delta^r$. Here, we claim:

Lemma 1. $y_0 = \langle g_0, \Omega \rangle$ *is non-vanishing at* $z = 0$.

Proof. Since the monodromy operator $\mathcal{T}_i$ preserving the intersection pairing $\langle\, , \rangle$, the nilpotent operator N is skew-symmetric with respect to the pairing. i.e. For $a, b \in H^3(\check{X}, \mathbb{C})$, we have:

$$\langle N(a), b \rangle + \langle a, N(b) \rangle = 0.$$

Next, $W_0 = \mathrm{im}(N^3)$, so if $x \in W_4$, $N^3 y \in W_0$,

$$\langle x, N^3(y) \rangle = -\langle N^3(x), y \rangle = 0$$

since $N^3(W_4) = 0$. Conversely, if $x \in W_0^{\perp}$, then $N^3 x = 0$. Thus, we have proved $W_0^{\perp} = W_4$. Similarly, one can prove $W_2 = W_2^{\perp}$.

Now, if $y_0(0) = 0$, then $\langle g_0(0), \Omega(0) \rangle = 0$ in the fiber of $\overline{\mathcal{H}}_{\check{X}}$ at $z = 0$. Then since $\Omega(0) \in \overline{\mathcal{F}^3}$, $g_0 \in W_0$. Hence, from above, we see that $\Omega(0) \in F^3_{\mathrm{lim}} \cap W_4 = 0$. However, $\Omega(z) \neq 0$ for any $z \neq 0$. $\square$

Hence, we can normalize Ω by Ω/y_0 so that $\int_{\gamma_0} \Omega = 1$. By Griffths' transversality and that y_0 is a holomorphic period on S, the effect of normalization on Yukawa coupling is given by

$$\int_{\check{X}} \frac{\Omega}{y_0} \wedge \nabla_{\delta_i} \nabla_{\delta_j} \nabla_{\delta_k} \frac{\Omega}{y_0} = \frac{1}{{y_0}^2} \int_{\check{X}} \Omega \wedge \nabla_{\delta_i} \nabla_{\delta_j} \nabla_{\delta_k} \Omega = Y/{y_0}^2.$$

We say the holomorphic 3-form $\Omega(s)$ is *normalized* if $\langle g_0, \Omega \rangle = 1$

Next, we pick the canonical coordinate q_i on S as following. In a neighborhood of LCSL point $z = 0$, we have the Hodge filtration $\mathcal{F}^\bullet$ of $\mathcal{H}$, and weight filtration $W_\bullet$ gives a filtration $\mathcal{W}_\bullet$ of $\mathcal{H}$. Recall that the condition of LCSL point implies $(W, \mathcal{F}_{\mathrm{lim}})$ is Hodge-Tate type. We choose a splitting $I^{p,p} = \mathcal{W}_{2p} \cap \mathcal{F}^p$. Then $I^{p,p} \cong Gr_{2p}(\mathcal{W}_\bullet)$. Also, since $N(\mathcal{W}_{2p}) \subset \mathcal{W}_{2p-2} = \mathcal{W}_{2p-1}$, N acts trivially on $Gr_{2p}(\mathcal{W}_\bullet) = \mathcal{W}_{2p}/\mathcal{W}_{2p-1}$. Therefore, the induced connection $\nabla^{\mathcal{W}}_{2p}$ of Gauss-Manin connection on $Gr_{2p}(\mathcal{W}_\bullet)$ has no monodromy. Thus, $\nabla^{\mathcal{W}}_{2p}$ can extend trivially to $z = 0$. We can choose integral valued $\nabla^{\mathcal{W}}_{2p}$-flat frame on Δ^r: e^0 of $I^{0,0}$, $e^1, \ldots, e^r$ of $I^{1,1}$, $e_1, \ldots e_r$ of $I^{2,2}$, and e_0 of $I^{3,3}$ such that, for $1 \leq i, j \leq r$,

$$\langle e_i, e^j \rangle = \delta_{ij},$$

and $\langle e_0, e^0 \rangle = -1$.

Now, let $\delta_i = 2\pi\sqrt{-1} z_i \partial/\partial z_i$ and $\nabla_i := \nabla_{\delta_i}$, we have $\nabla_i(e_0) \in \mathcal{F}^2$ by Griffths' transversality. Also, at $z = 0$, notice that $\nabla_i|_{z=0} = N_i$, and hence $\nabla_i(e^0)(0) = N_i(e_0(0)) \in W_4$.

Thus, $\nabla_i(e_0) \in I^{2,2}$. Similarly, $\nabla_i e_j \in I^{1,1}$, $\nabla_i e^j \in I^{0,0}$. Thus, we can let

$$\begin{aligned}
\nabla_i e_0 &= \sum_{j=1}^{r} a_{ij} e_j, \\
\nabla_i e_j &= \sum_{k=1}^{r} b_{ijk} e^k, \\
\nabla_i e^j &= c_{ij} e^0, \\
\nabla_i e^0 &= 0,
\end{aligned}$$

for some holomorphic functions a_{ij}, b_{ijk}, c_{ij}. (Since ∇ has regular singular point). Since $\langle e_0, e^k \rangle = 0$, we have

$$\begin{aligned}
0 = \delta_i \langle e_0, e^k \rangle &= \langle \nabla_i(e_0), e^k \rangle + \langle e_0, \nabla_i(e^k) \rangle \\
&= \langle \sum_{j=1}^{r} a_{ij} e_j, e^k \rangle + \langle e_0, c_{ik} e^0 \rangle \\
&= a_{ik} - c_{ik}.
\end{aligned}$$

Hence, $a_{ij} = c_{ij}$. We claim that we can take e_0 to be the normalized 3-form. Notice that given the normalized 3-form Ω, we write $\Omega = f(z_1, \ldots, z_r) e_0$, for some holomorphic function f. Then

$$\nabla^i_{\delta_j} \Omega \equiv \delta_j(f) e_0, \quad \mod e_i, e^j, e^0.$$

Therefore, f also satisfies Picard-Fuchs equations. Since at LCSL point, there is only one holomorphic solution $\langle g_0, \Omega \rangle = 1$. Thus, $f(z_1, \ldots, z_r)$ is some constant f. Since $e^0 \in W_0$, we can take e^0 to be g_0. Then

$$1 = \langle g_0, \Omega \rangle = \langle e^0, f e_0 \rangle = -f \langle e^0, e_0 \rangle = f.$$

Next, we compute

$$\nabla_i(e^j)(0) = N_i(e^j) = m_{ij} e^0,$$

where m_{ij} is the invertible matrix occurring in the definition of LCSL point. Hence, we know that $a_{ij}(0) = c_{ij}(0) = m_{ij}$. For $i = 1, \ldots, r$, we then define

$$t_i := \sum_{j=1}^{r} \int_{z_j} \frac{a_{ij}(z) dz_j}{z_j}.$$

Then using residue, we compute

$$\left(z_i \frac{\partial t_i}{\partial z_j} \right)(0) = m_{ij}.$$

Hence, t_i defines a coordinate around 0. Let $q_i = \exp(t_i)$, then we have:

$$z_i \frac{\partial}{\partial z_i} = \sum_{j=1}^{r} z_i \frac{\partial q_j}{\partial z_i} \frac{\partial}{\partial q_j} = \sum_{j=1}^{r} a_{ij} q_j \frac{\partial}{\partial q_j}$$

Redefining δ_i to be $2\pi\sqrt{-1}\partial/\partial q_i$ and $\nabla_i := \nabla_{\delta_i}$, we have

$$\begin{aligned}\nabla_i e_0 &= e_i,\\ \nabla_i e_j &= \sum_{k=1}^{r} Y'_{ijk} e^k,\\ \nabla_i e^j &= \delta_{ij} e^0,\\ \nabla_i e^0 &= 0,\end{aligned}\tag{2}$$

where Y'_{ijk} is the new function appearing in q_i-coordinate. We now identify Y'_{ijk} as Yukawa couplings in q_i-coordinate. Since e_0 is the normalized 3-form, we compute

$$\begin{aligned}\nabla_i\nabla_j\nabla_k(e_0) &= \nabla_i\nabla_j(e_k) = \nabla_i\left(\sum_{l=1}^{r} Y'_{jkl}e^l\right)\\ &= \sum_{l=1}^{r}\delta_i(Y'_{jkl})e^l + \sum_{l=1}^{r} Y'_{jkl}\delta_{il}e^0 = \sum_{l=1}^{r}\delta_i(Y'_{jkl})e^l + Y'_{jki}e^0.\end{aligned}$$

Then

$$Y_{ijk} = \langle e_0, \nabla_i\nabla_j\nabla_k(e_0)\rangle = -Y'_{jki}.$$

Under the canonical coordinate q_i and the normalized volume form Ω, we define the *normalized Yukawa coupling* by

$$Y_{ijk} = -\int_{\check{X}} \Omega \wedge \nabla_k\nabla_i\nabla_j\Omega.$$

We end this section by showing that the normalized Yukawa couplings are in fact the third derivatives of some potential function. We let variable u_i on U^r such that $q_j = \exp(2\pi\sqrt{-1}u_j)$ so that $\partial/\partial u_j = 2\pi\sqrt{-1}q_j\partial/\partial q_j = \delta_j$.

Proposition 1. *There exists a holomorphic function $\Phi^{GM}(u_1, \ldots, u_r)$ such that*

$$Y_{ijk} := \frac{1}{(2\pi\sqrt{-1})^3}\frac{\partial^3\Phi^{GM}}{\partial u_i\partial u_j\partial u_k},$$

for all i, j, k.

Proof. Since Gauss-Manin connection ∇ is flat, we have:

$$0 = \nabla^2(e_0) = \sum_{i,j,k=1}^{r} Y_{ijk}e^k \otimes dt_i \wedge du_j.$$

Therefore, one can see that

$$Y_{ijk} = Y_{jik}.\tag{3}$$

Also, for each $j = 1, \ldots, r$, one computes readily that

$$0 = \nabla^2(e_j) = \sum_{i,k=1}^{r} e^0 \otimes du_k \wedge dt_i + \sum_{i,k,l=1}^{r}\frac{\partial Y_{ijk}}{\partial u_l}e^k \otimes du_l \wedge du_i.$$

This implies

$$Y_{ijk} = Y_{kji} \tag{4}$$

$$\frac{\partial Y_{ijk}}{\partial u_l} = \frac{\partial Y_{ljk}}{\partial u_i} \tag{5}$$

From (3), (4), (5), we can construct a potential function Φ^{GM}. □

3. A-model: Kähler Moduli and Quantum Cohomology

3.1. **Comlexified Kähler Moduli.** Given a smooth Calabi-Yau 3-fold X with Kähler cone $\mathcal{K}(X)$, we define the complexified Kähler space of X by:

$$K_{\mathbb{C}}(X) := (H^2(X,\mathbb{R}) + \sqrt{-1}\mathcal{K}(X))/H^2(X,\mathbb{Z})$$

When $h^{1,1}(X) = 1$, then the Kähler cone $\mathcal{K}(X)$ is an open ray, and

$$K_{\mathbb{C}}(X) = (\mathbb{R} + \sqrt{-1}\mathbb{R}_{>0})/\mathbb{Z} = \mathbb{H}/\mathbb{Z},$$

where $\mathbb{Z}$ acting on upper half-plane $\mathbb{H}$ by $z \mapsto z+1$. Hence, the complexified Kähler space of X can be identified with the punctured unit disk Δ^* with universal covering $\mathbb{H} \to \Delta^*$ by $z \mapsto q = e^{2\pi\sqrt{-1}z}$.

For $h^{1,1}(X) = r > 1$, since $h^{2,0}(X) = 0$, $H^2(X,\mathbb{C}) = H^{1,1}(X)$, and $\mathcal{K}(X)$ is an open cone in $H^{1,1}(X) \cap H^2(X,\mathbb{R})$, we can take a basis $T_1, \ldots, T_r$ of $H^2(X,\mathbb{Z})_{\text{free}}$ with $T_i \in \overline{\mathcal{K}(X)}$. With this choice of basis, called a *framing*, we obtain a simplicial cone $\Sigma := \{\sum_{i=1}^r u_i T_i : t_i > 0\}$. We restrict ourselves to the subspace $K_{\mathbb{C},\Sigma}(X) = (H^2(X,\mathbb{R}) + \sqrt{-1}\Sigma)/H^2(X,\mathbb{Z})_{\text{free}}$, and we can identify it to $(\Delta^*)^r$ via the map

$$\sum_{i=1}^r u_i T_i \mapsto (q_1, \ldots, q_r) = (e^{2\pi\sqrt{-1}u_1}, \ldots, e^{2\pi\sqrt{-1}u_r}).$$

Thus, we have a partial compactification $(\Delta^*)^r \subset \Delta^r$, and we call the origin $0 \in \Delta^r$ a *large radius limit point.*

3.2. **Gromov-Witten Invariants and Quantum Cohomology for Calabi-Yau 3-folds.** First of all, for a Calabi-Yau 3-fold X, we have the following simple observation on its Gromov-Witten invariants.

Lemma 2. *The genus 0 Gromov-Witten invariants on X are determined by intersection numbers and $N_\beta := \langle\ \rangle_{0,0,\beta} = \int_{[\overline{M}_{0,0}(X,\beta)]^{vir}} 1$*

Proof. For a smooth Calabi-Yau 3-fold X, and an effective class $\beta \in H_2(X,\mathbb{Z})$, the virtual dimension of the moduli space of genus 0, n-pointed stable maps to X representing the class β is

$$\text{v.}\dim_{\mathbb{C}} \overline{M}_{0,n}(X,\beta) = \int_\beta c(X) + (\dim X - 3)(1-g) + n = n.$$

Therefore, for $\beta \neq 0$, given homogeneous elements $a_1, \ldots, a_n \in H^*(X,\mathbb{C})$, the Gromov-Witten invariants $\langle a_1, \ldots, a_n\rangle_{0,n,\beta}$ is non-zero only if $a_1, \ldots, a_n \in H^2(X,\mathbb{C})$ or one of

$a_i = 1$. In the former case, by divisor axiom, we have:

$$\langle a_1, \dots, a_n \rangle_{0,n,\beta} = \left(\prod_{i=1}^{n} \int_\beta a_i \right) \cdot \langle \ \rangle_{0,0,\beta}.$$

While in the second case, fundamental class axiom shows that it can only be zero. On the other hand, if $\beta = 0$, then by point mapping axiom, the only non-zero case is $n = 3$, which is

$$\langle a_1, a_2, a_3 \rangle_{0.3.0} = \int_X a_1 \cup a_2 \cup a_3.$$

□

From the above lemma, we see that non-classical information occurs only when the insertions are in $H^2(X, \mathbb{C})$ in both big and small quantum cohomology. Now, we extend the basis of $H^2(X, \mathbb{C})$ in the previous section to a basis $T_0 = 1, T_1, \dots T_m$ of $H^*(X, \mathbb{C})$ with dual basis $\{T^i\}_{i=0}^m$ relative to the intersection pairing g, and associated variables $\{t_i\}_{i=0}^m$ Then, for $a, b \in H^*(X, \mathbb{C})$, their small quantum product is

$$a *_s b := \sum_{k=0}^{r} \sum_{\beta \in \mathrm{NE}(X)} \langle a, b, T_k \rangle_{0,3,\beta} T^k q^\beta,$$

where $\mathrm{NE}(X)$ is the cone of effective 1-cycle in X, q^β is the Novikov variable. Hence, for $a, b, c \in H^2(X, \mathbb{C})$, the three-point function or A-model correlation function becomes:

$$\langle a, b, c \rangle := \sum_\beta \langle a, b, c \rangle_{0,3,\beta} q^\beta = \int_X a \cup b \cup c + \sum_{\beta \neq 0} N_\beta q^\beta \int_\beta a \int_\beta b \int_\beta c$$

Now, if we arrange that $T_1, \dots, T_r$ is our chosen basis of $H^2(X, \mathbb{C})$ in previous section, and set $D = \sum_{i=1}^r t_i T_i$. On Calabi-Yau 3-folds, the genus 0 Gromov-Witten potential is given by

$$\Phi^{GW} = \Phi_{\text{classical}} + \sum_{\beta \neq 0} N_\beta e^{\int_\beta D} q^\beta, \tag{6}$$

where $\Phi_{\text{classical}}$ is the term involving cup product only. Then for $T_i, T_j \in H^*(X, \mathbb{C})$, the big quantum product

$$\begin{aligned} T_i * T_j &= \sum_k \frac{\partial^3 \Phi^{GW}}{\partial t_i t_j t_k} T^k \\ &= \left(\int_X T_i \cup T_j \cup T_k + \sum_{\beta \neq 0} N_\beta \int_\beta T_i \int_\beta T_j \int_\beta T_k e^{\int_\beta D} q^\beta \right) T^k. \end{aligned}$$

Remark 3. We regard q^β as formal variable so far. In the following discussion, we regard q^β as a function on complexified Kähler moduli space as following. Let $\omega = H^2(X, \mathbb{R}) + \sqrt{-1} \sum_{i=1}^r u_i T_i$ be a chosen complexified Kähler class on X, then we define

$$q^\beta = e^{2\pi\sqrt{-1} \int_\beta \omega} = q_1^{u_1} \dots q_r^{u_r},$$

where $q_i = e^{-\int_\beta T_i}$. Also, as in [5], we **assume** that small quantum product on Calabi-Yau manifolds converge for ω has sufficiently large imaginary part. i.e., ω is close to large radius limit point $q = 0$. Also, we note that in the above expression of (6), we have the exponential term

$$e^{\int_\beta D} q^\beta = e^{\int_\beta (D+2\pi\omega)}.$$

If we set $\omega = 0$ and set

$$q^\beta = e^{\int_\beta D} = \exp\left(\int_\beta \sum_{i=1}^r t_i T_i\right),$$

then

$$T_i * T_j = \sum_\beta \langle T_i, T_j, T_k\rangle_{0,3,\beta} T^k q^\beta.$$

Therefore, the small quantum product $T_i *_s T_j$ agrees with $T_i * T_j$ provided we set $\omega = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^r t_i T_i$ in the definition of small quantum product. We relate two sets of coordinates u_i and t_i by $u_i = \frac{1}{2\pi\sqrt{-1}} t_i$.

A final comment is that on Calabi-Yau n-folds, the small quantum product is compatible with Hodge decomposition. i.e., for $a \in H^{p,q}(X), b \in H^{r,s}(X)$, $a *_s b \in H^{p+r,q+s}(X)$. In particular, for $a \in H^k(X, \mathbb{C}), b \in H^l(X, \mathbb{C})$, we have $\deg(a *_s b) = k+l$. To see this, notice that $a *_s b = \sum_\beta \langle a, b, T_i\rangle_{0,3,\beta} T^i q^\beta$ and if $T^i \in H^{t,w}(X)$, then$T_i \in H^{n-t,n-w}(X)$. Hence, it suffices to prove that $\langle a, b, c\rangle_{0,3,\beta} = 0$ unless $c \in H^{n-(p+r),n-(q+s)}$. By definition,

$$\langle a, b, c\rangle = \int_{\overline{M}_{0,3}(X,\beta)^{\mathrm{vir}}} e_1^* a \cup e_2^* b \cup e_3^* c = \int_{e_*[\overline{M}_{0,3}(X,\beta)^{\mathrm{vir}}]} a \otimes b \otimes c,$$

where $e : \overline{M}_{0,3}(X, \beta) \to X^3$ is the evaluation map. Since $\mathrm{v.dim}\, \overline{M}_{0,3}(X, \beta) = \dim X = n$, $e_*[\overline{M}_{0,3}(X, \beta)^{\mathrm{vir}}] \in H_{2n}(X^3)$ is a homology class of algebraic cycle. Hence, the above integral is vanishing unless $a \otimes b \otimes c$ has type (n, n). Therefore, this implies

Proposition 2. *For a Calabi-Yau n-fold X, $\bigoplus_{i=0}^n H^{i,i}(X)$ is a subring of $H^*(X, \mathbb{C})$ with respect to the small quantum product.*

3.3. **Dubrovin Connection and J-function.** Consider the *Dubrovin connection* defined on the trivial bundle $\mathcal{H}_X$ over $(\Delta^*)^r$ with fiber $H^*(X, \mathbb{C})$:

$$\nabla^z_{\frac{\partial}{\partial t_i}} = \frac{\partial}{\partial t_i} + \frac{1}{z}(T_i *\ ),$$

where z is a parameter $z \in \mathbb{C}^*$. The flatness of the connection follows from WDVV equation (though it is trivial in the case of Calabi-Yau 3-fold). On the other hand, we define *A-model connection* ∇^A on $\mathcal{H}_X$ by

$$\nabla^A_{\partial/\partial u_i} T_j = T_i *_s T_j,$$

for $i = 1, \ldots, r$, $j = 1, \ldots, m$. If we restrict Dubrovin connection ∇^z to $H^2(X, \mathbb{C})$, as remark 3, we have:

$$\nabla^z_{\partial/\partial t_i} T_j = \frac{1}{z} \sum_\beta \langle T_i, T_j, T_k\rangle T^k q^\beta = \frac{1}{z} T_i *_s T_j,$$

provided that the ω in the $q^\beta = \exp(2\pi\sqrt{-1}\int_\beta \omega)$ is set to be

$$\omega = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{r} t_i T_i.$$

Since A-model connection is defined using u_i coordinate and $u_i = \frac{1}{2\pi\sqrt{-1}} t_i$, we have

$$\frac{\partial}{\partial u_i} = 2\pi\sqrt{-1}\frac{\partial}{\partial t_i}.$$

Then this gives

$$\nabla^A_{\partial/\partial t_i} T_j = \frac{1}{2\pi\sqrt{-1}} T_i *_s T_j,$$

and hence we have

$$\nabla^A = \nabla^z,$$

for $z = 2\pi\sqrt{-1}$.

In terms of q-coordinate, $q_j = e^{2\pi\sqrt{-1}u_j}$,

$$\frac{\partial}{\partial u_j} = 2\pi\sqrt{-1} q_j \frac{\partial}{\partial q_j}$$

Hence, the A-model connection becomes:

$$\nabla^A_{\frac{\partial}{\partial q_j}} T_k = \frac{1}{2\pi\sqrt{-1} q_j} T_j *_s T_k.$$

We have the following.

Proposition 3. *The A-model connection ∇ of Calabi-Yau 3-fold X has regular singular point along $\Delta^r \setminus (\Delta^*)^r$.*

Now, to find the flat section of Dubrovin connection, we introduce the J-function. For $t = \sum_i t_i T_i \in H^*(X, \mathbb{C})$, we define

$$J(t, z^{-1}) := 1 + \frac{t}{z} + \sum_{n,\beta \neq 0} \left\langle \frac{T_b}{z(z-\psi)}, t^n \right\rangle_{0,\beta} q^\beta T^b = 1 + J_b T^b,$$

where

$$J_b = \frac{1}{z}\sum_{i=1}^{r} g_{ib} t_i + \sum_{n,\beta\neq 0} \frac{1}{n!} \left\langle \frac{T_b}{z(z-\psi)}, t^n \right\rangle_{0,\beta} q^\beta.$$

Proposition 4. *The matrix*

$$\Psi_{ab} := z\frac{\partial}{\partial t_a} J_b = g_{ab} + \sum_{n,\beta} \left\langle T_a, \frac{T_b}{z(z-\psi)}, t^n \right\rangle q^\beta$$

gives the fundamental solution of Dubrovin connection. More precisely, let $s_b := \Psi_{ab} T^a$, we have:

$$\frac{\partial}{\partial t_i} s_b = z^{-1} T_i * s_b$$

For $\gamma_i \in H^*(X, \mathbb{C})$, we define the genus g coupling of descendent Gromov-Witten invariants by:

$$\langle\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n \rangle\rangle_g := \sum_{k=1}^{\infty} \sum_{\beta} \frac{1}{k!} \left\langle \tau_{d_1}\gamma_1, \dots \tau_{d_n}\gamma_n, \gamma^k \right\rangle_{g,\beta} q^\beta$$

The proposition follows from the following important relation on the genus 0 couplings:

Theorem 1 (Topological Recursion Relation)**.** *For $d \geq 0$, and for $i, j, k \in 0, \dots, m$, we have:*

$$\langle\langle \tau_{d+1}T_i, T_j, T_k \rangle\rangle_0 = \langle\langle \tau_d T_i, T_a \rangle\rangle_0 \langle\langle T^a, T_j, T_k \rangle\rangle_0$$

For a proof of genus 0 topological recursion relation, we refer to [13], Chap. VI Theorem 6.2.

Proof of the Proposition. First, notice that:

$$\frac{\partial}{\partial t_i} \langle\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n \rangle\rangle_0 = \langle\langle T_i, \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n \rangle\rangle_0.$$

Hence, direct computation shows

$$\frac{\partial s_b}{\partial t_i} = \sum_{k \geq 0} z^{-(k+1)} \langle\langle T_i, \tau_k T_b, T_a \rangle\rangle_0 T^a.$$

On the other hand,

$$\begin{aligned}
\frac{1}{z} T_i * s_b &= \frac{1}{z} T_i * T_b + \sum_{k \geq 0} z^{-(k+2)} \langle\langle \tau_k T_b, T_a \rangle\rangle_0 (T_i * T^a) \\
&= \frac{1}{z} T_i * T_b + \sum_{k \geq 0} z^{-(k+2)} \langle\langle \tau_k T_b, T_a \rangle\rangle_0 \langle\langle T_i, T^a, T_s \rangle\rangle_0 T^s \\
&= \frac{1}{z} \langle\langle T_i, T_b, T_a \rangle\rangle T^a + \sum_{k \geq 0} z^{-(k+2)} \langle\langle \tau_k T_b, T_s \rangle\rangle_0 \langle\langle T_i, T^s, T_a \rangle\rangle_0 T^a \\
&= \frac{1}{z} \langle\langle T_i, T_b, T_a \rangle\rangle T^a + \sum_{k \geq 1} z^{-(k+1)} \langle\langle \tau_k T_b, T_i, T_a \rangle\rangle T^a \\
&= \sum_{k \geq 0} z^{-(k+1)} \langle\langle T_i, \tau_k T_b, T_a \rangle\rangle_0 T^a,
\end{aligned}$$

where the last equality use the topological recursion relation. □

Since Dubrovin connection coincides with A-model connection when restricting to $H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C})$ and set $z = 2\pi\sqrt{-1}$, flat sections for Dubrovin connection will be the flat sections for A-model connection. We will fix this value. Using axioms of Gromov-Witten invariants, when we restrict the variable to $t = t_0 + D = t_0 + \sum_{i=1}^r t_i T_i \in H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C})$, the flat sections become ([5], Prop 10.2.3)

$$s_b|_{t \in H^0 \oplus H^2} = e^{\frac{t_0 + D}{z}} \cup T_b + e^{\frac{t_0}{z}} \sum_{\beta \neq 0} q^\beta \langle \frac{e^{\frac{D}{z}} \cup T_b}{z - \psi}, T_a \rangle_{0,\beta} T^a,$$

where $q^\beta = e^{\int_\beta D}$. More generally, for $T \in H^*(X, \mathbb{C})$, the flat section of A-model connection ∇ is given by

$$s(T) := e^{D/z} \cup T + \sum_{\beta \neq 0} q^\beta \langle \frac{e^{D/z} \cup T}{z - \psi}, T_a \rangle_{0,\beta} T^a. \tag{7}$$

3.4. **A-Variation of Hodge Structure.** We have defined the A-model connection ∇^A in previous section. In this section, we will define the remaining ingredients constituting a VHS on $(\Delta^*)^r$ and its degeneration on the partial compactification Δ^r.

We first determine the monodromy of ∇^A. We know that (7) gives the flat section of ∇^A when we set $z = 2\pi\sqrt{-1}$. Let $\mathcal{T}_j$ be the monodromy operator about the j-th direction,.

Proposition 5. *The monodromy operators acts on flat sections of the A-model connection as follows:*

$$\mathcal{T}_j(s(T)) = s(e^{T_j} \cup T)$$

Proof. Since $q_j = e^{t_j}$, $\mathcal{T}_j$ takes t_j to $t_j + 2\pi\sqrt{-1}$. Since $z = 2\pi\sqrt{-1}$, we can write $t_j \mapsto t_j + z$. Then the effect of this on (7) comes from $D = \sum_{i=1}^r t_i T_i \mapsto D_i + zT_j$. For $q^\beta = e^{\int_b etaD}$, this has no effect since T_i and β are integral class. Therefore, the only effect is given by

$$e^{D/z} \mapsto e^{(D+zT_j)/z} = e^{D/z} \cup e^{T_j}.$$

$\square$

Taking logarithm, we get $N_j := \log(\mathcal{T}_j)$ is cup product with T_j from the left. Hence, N_j is nilpotent. One can also prove this by looking at the residue matrix of ∇^A at $q = 0$ along a line q_j. Since $q_j^\beta \to 0$ for $\beta \neq 0$, the (l, s)-entry of residue matrix of $\nabla^A_{\partial/\partial q_j}$ at $q_j = 0$ is

$$\frac{1}{2\pi\sqrt{-1}} \sum_l \langle T_j, T_k, T_l \rangle_{0,3,0} g^{ls} = \frac{1}{2\pi\sqrt{-1}} \sum_l g^{ls} \int_X T_j \cup T_k \cup T_l$$

Hence, the nilpotent operator is agian obtained from cup product with T_j.

Next, as in the case of B-model, if $s(T)$ is a flat section of ∇^A, then

$$\tilde{s}(T) = \exp(\frac{-1}{2\pi\sqrt{-1}} \sum_{j=1}^r log(q_j) N_j) s(T)$$

extends naturally to the canonical extension $\overline{\mathcal{H}}_X$ over Δ^r. Then

$$\begin{aligned}\tilde{s}(T) &= \exp(-z^{-1}\sum_j t_j N_j)s(T)\\ &= s(\exp(z{-}1\sum_j t_j(-T_j))\cup T)\\ &= s(e^{-D/z}\cup T)\\ &= T+\sum_{\beta\neq 0}\sum_{j=0}^{m} q^\beta\langle\frac{T}{z-\psi},T_j\rangle_{0,\beta}T^j.\end{aligned}$$

Since β is effective and $T_1,\dots,T_r$ lie in the closure of Kähler cone, we have $\int_\beta T_i\geq 0$. Therefore, q^β has non-negative exponents in q_i, and hence $\tilde{s}$ indeed extends holomorphically to $q=0$. Moreover, since $T_1,\dots,T_r$ forms a basis of $H^2(X,\mathbb{C})$, for $\beta\neq 0$, there exists some i such that $\int_\beta T_i>0$. Therefore, $\tilde{s}(T)(0)=T$. Also, by expanding the formula of $\tilde{s}(T)$

$$\tilde{s}(T)=T+\sum_{\beta\neq 0}\sum_{j=0}^{m}\sum_{n=0}^{\infty}q^\beta z^{-(n+1)}\langle\tau_n T,T_j\rangle_{0,\beta}T^j,$$

we find that for each summand to be non-vanishing, we must have $\deg T^j=\deg T+2n+2>\deg T$. In summary, we have proved:

Proposition 6. *For each homogeneous element $T\in H^*(X,\mathbb{C})$, the section $\tilde{s}(T)$ satisfies $\tilde{s}(T)=T+$(higher degree term) and $\tilde{s}(T)(0)=T$. Also, the nilpotent operator N_j acting on $\tilde{s}(T_a)$ is just the cup product of T_j on T_a on the left.*

Once we know the monodromy, we can determine the monodromy weight filtration for the A-model connection.

Proposition 7. *For a Calabi-Yau 3-fold X, the monodromy weight filtration $W_\bullet$ of ∇^A at $0\in\Delta^r$ is given by*

$$W_i=\bigoplus_{j\geq 6-i}H^j(X,\mathbb{C})$$

Proof. For any $N=\sum_{j=1}^r a_jN_j$, where $N_j=\log\mathcal{T}_j$ and $a_j>0$. We know that the action of N_j on the fiber over 0 is just $T_j\cup$. Therefore, N is the cup product with the Kähler class $\omega=\sum_{i=1}^r a_jT_j$ on the fiber over 0. Then for $\alpha\in\bigoplus_{j\geq 6-i}H^j(X,\mathbb{C})$, then $\omega\cup\alpha\in\bigoplus_{j\geq 6-i}H^{j+2}(X,\mathbb{C})=W_{i-2}$. Also, the hard Lefschetz theorem implies

$$(\omega\cup)^k: H^{3-k}(X,\mathbb{C})=Gr_{3-k}(W_\bullet)\cong H^{3+k}=Gr_{3+k}(W_\bullet).$$

Therefore, by the discussion in sec.2.1, we see that W_i is uniquely determined. □

Since $\tilde{s}_k(0)=T_k$, $\tilde{s}_k\in W_i$ if $\deg(T_k)\geq 6-i$. Away from 0, the remaining terms of $\tilde{s}_k$ are of higher degree. By definition of W_i, we have the following immediate corollary.

Corollary 1. *The monodromy weight filtration $\mathcal{W}_\bullet$ of ∇^A over Δ^r is given by the trivial bundle*

$$\mathcal{W}_i=\oplus_{j\geq 6-i}H^j(X,\mathbb{C})\times\Delta^r.$$

Now, since X is a Calabi-Yau 3-fold with $h^{1,1}(X) = r$. We see that $\dim W_0 = \dim W_1 = \dim H^6(X, \mathbb{C}) = 1$ and $W_2 = H^6(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$ has dimension $r + 1$. Moreover, if $[pt]$ is the generator of $H^6(X, \mathbb{C})$, $g_1, \ldots, g_r \in H^4(X, \mathbb{C})$ are dual of $T_1, \ldots, T_r$, then

$$N_i(g_j) = T_i \cup g_j = (\int_X T_i \cup g_j)[pt] = \delta_{ij}[pt].$$

As a result, we have:

Proposition 8. *The A-model connection ∇^A has mximally unipotent monodromy at $0 \in \Delta^r$.*

Next, we define the Hodge filtration for A-model connection. $F^p = \bigoplus_b \bigoplus_{a \leq 3-p} H^{a,b}(X)$ gives a decreasing filtration on $H^*(X, \mathbb{C})$. We have an obvious bundle version on $\mathcal{H}_X$ given by

$$\mathcal{F}^p = F^p \times (\Delta^*)^r.$$

Note that given $\alpha \in F^p$, recarded as a section on $\mathcal{F}^p$, we have:

$$\nabla^A_{\partial/\partial u_i} \alpha = T_i *_s \alpha \in H^{a+1,b+1}.$$

Then $\nabla^A_{\partial/\partial u_i} \alpha \in \mathcal{F}^{p-1}$. Thus, we have proved:

Proposition 9. *The filtration $\mathcal{F}^\bullet$ satisfies Griffths' transversality with resepct to A-model connection ∇^A.*

We now a define polarization S on $\mathcal{H}_X$. Given $a \in H^k(X, \mathbb{C}), b \in H^l(X, \mathbb{C})$, we define

$$S(a, b) = \begin{cases} (-1)^{k(k+1)/2} \int_X a \cup b & k + l = 2 \dim X = 6 \\ 0 & k + l \neq 2 \dim X = 6. \end{cases}$$

We regard S is defined on $\mathcal{H}_X$ when thinking a, b as sections of $\mathcal{H}_X$. From definition, $S(a, b) = -S(b, a)$. Moreover, we have

Lemma 3. *(1) S is flat with respect to ∇^A, i,e,*

$$\frac{\partial}{\partial u_i} S(a, b) = S(\nabla^A_{\partial/\partial u_i}(a), b) + S(a, \nabla^A_{\partial/\partial u_i} b)$$

(2) $S(\mathcal{F}^p, \mathcal{F}^{3-p+1}) = 0$.

Proof. For $a = T_j, b = T_l$, it suffices to prove

$$S(T_i *_s T_j, T_l) + S(T_j, T_i *_s T_l) = 0.$$

We may assume that $\deg(T_j) = k$ and $\deg(T_l) = 6 - k - 2$. Then $\deg(T_i *_s T_j) = k + 2$.

$$S(T_i *_s T_j, T_l) = (-1)^{(k+2)(k+3)/2} \int_X (T_i *_s T_j) \cup T_l = -(-1)^{(k+1)(k+2)/2} \int_X (T_i *_s T_j) *_s T_l,$$

, where $\int_X (T_i *_s T_j) \cup T_l = \int_X (T_i *_s T_j) *_s T_l$ is directly from degree condition. Similarly,

$$S(T_j, T_i *_s T_l) = (-1)^{(k+1)(k+2)/2} \int_X (T_j *_s T_i) *_s T_l = (-1)^{(k+1)(k+2)/2} \int_X (T_i *_s T_j) *_s T_l.$$

For the second part, just observe that for $a \in F^p$, $b \in F^{3-p+1}$, $a \cup b$ is impossible of type $(3, 3)$. $\square$

Now, by lemma 3 (2), we see that $F^p = \bigoplus_b \bigoplus_{a\leq 3-p} H^{a,b}(X)$ lies in $\check{D}$. (cf. the notation in sec.2.1) Also, we have defined the monodromy weight filtration $W_k = \bigoplus_{k\geq 6-i} H^i(X,\mathbb{C})$. Moreover, at 0, the monodromy logarithm is just cup product with T_j. As we have seen in Example 1, for any N lying in the cone spanned by $N_1,\ldots,N_r$, $(W_\bullet, F^\bullet)$ forms a mixed Hodge structure on $H^*(X,\mathbb{C})$ polarized by N with respect to intersection form S.

From the discussion in sec.2.1, we consider the filtration $\mathcal{F}_{\text{nil}} := \exp(\sum_i u_i N_i)\cdot F$, which is a nilpotent orbit. Hence, this gives a variantaion of Hodge structure $(\mathcal{H}_X, \nabla^A, \mathcal{F}_{\text{nil}})$ in a neighborhood of $q = 0$. On the other hand, $\mathcal{F}$ coincdes with $\mathcal{F}_{\text{nil}}$ at 0 and we have proved that $\mathcal{F}$ satisfying Griffths' transversality and first Hodge-Riemann bilinear relation, by [7] sec.2.3, $(\mathcal{H}_X, \nabla^A, \mathcal{F})$ is a variation of Hodge structure in a neighborhood of 0.

Recall in sec.2.2 that near the LCSL point, the mixed Hodge structure is of Hodge-Tate type. On the other hand, as we have seen in exmaple 1, the mixed Hodge structure of A-model variation at 0 is of Hodge-Tate when we restrict to $\bigoplus_p H^{p,p}$. We let $\mathcal{H}^{\text{middle}}$ by the trivial bundle $\bigoplus_{p=0}^3 H^{p,p}(X)$. Moreover, by proposition 2, we have seen that $\bigoplus_{p=0}^3 H^{p,p}(X)$ is a subring of $H^*(X)$ with respect to the small quantum product. Therefore, we can restrict A-model connection ∇^A to $\mathcal{H}^{\text{middle}}$, which we denote ∇^{middle}. Also, we still denote $\mathcal{F}$ and S to be the induced filtration and polarization on $\mathcal{H}^{\text{middle}}$. Previous discussion shows that the *middle A-variation of Hodge structure* or *A-variation*

$$(\mathcal{H}^{\text{middle}}, \nabla^{\text{middle}}, \mathcal{F}, S)$$

is a variation of polarized Hodge structure on a neighborhood of 0 in $(\Delta^*)^r$.

4. Mirror Symmetry

4.1. **Mirror Map.** Recall that at LCSL point $0 \in \Delta^r \subset \overline{\mathcal{M}}(X)$, the monodromy weight filtration W_0, W_2 have dimension $1, r+1$, resepctively. Let $g_0, \ldots g_r$ be a basis of W_2 with g_0 spanning W_0. Define the matrix (m_{ij}) by $N_i(g_j) = m_{ij} g_0$ with inverse m^{ij}. With above notation, we define *mirror map* $M : (z_1,\ldots,z_r) \mapsto (q_1,\ldots,q_r)$ by:

$$\frac{1}{2\pi\sqrt{-1}} \log q_k = \frac{1}{\langle g_0, \Omega\rangle} \sum_{j=1}^r \langle g_j, \Omega\rangle m^{jk} \tag{8}$$

If we set

$$y_k = \sum_{j=1}^r \langle g_j, \Omega\rangle m^{jk},$$

then notice that $N_j(\sum_{i=1}^r g_i m^{ik}) = \sum_{i=1}^r m_{ji} m^{ik} = \delta_{jk}$. Therefore, $y_k = \langle \sum_{j=1}^r g_k m^{jk}, \Omega\rangle$ is a solution of Picard-Fuchs equations which is holomorphic at general points of D_j for $j \neq k$ and has a logarithmic pole along D_k. Hence, we have

$$y_k = y_0 \frac{\log z_k}{2\pi\sqrt{-1}} + \tilde{y}_k,$$

where $\tilde{y}_k$ is holomorphic at $z = 0$. Thus, we can rewrite (8) as

$$\frac{1}{2\pi\sqrt{-1}} \log q_k = \frac{y_k}{y_0}$$

Hecne, we have:

$$q_k = z_k \exp(2\pi\sqrt{-1})\frac{\tilde{y}_k}{y_0}). \tag{9}$$

We conclude this section by showing that the canonical coordinate we found in sec.2.3 is exactly the mirror map. Recall that we have found a canonical cooordinate $q_1, \ldots, q_r$ on $(\Delta^*)^r$ and local sections e_0, e_i, e^i, e^0, where e_0 is the normalized 3-form Ω, e^0 is the integral generator g_0 of $\mathcal{W}_0$, and $e^1, \ldots e^r$ are sections of $\mathcal{W}_2$, such that the Gauss-Manin connection is given by

$$\begin{aligned}
\nabla_i e_0 &= e_i,\\
\nabla_i e_j &= \sum_{k=1}^{r} Y'_{ijk} e^k,\\
\nabla_i e^j &= \delta_{ij} e^0,\\
\nabla_i e^0 &= 0,
\end{aligned}$$

where $\nabla_i := \nabla_{2\pi\sqrt{-1}q_i\partial/\partial q_i}$. Now, if we write u_j as the coordinate on upper half-space U^r with $q_j = \exp(2\pi\sqrt{-1}u_j)$, and for $j = 1, \ldots, r$, let

$$g_j = -e^j + u_j e^0$$

Then $\nabla_i g_j = -\delta_{ij} e^0 + \delta_{ij} e_0 = 0$, and hence g_j are flat section. Note that in sec.2.1, we use t_j to denote coordinates on upper half-spaces. However, in accordance to the coordinates on Kähler moduli space introduced in sec.3.1 , we change to u_j. Furthermore, the monodromy operator of i-th direction $\mathcal{T}_i$ takes u_j to $u + \delta_{ij}$. This implies that $\mathcal{T}_i(g_j) = g_j + \delta_{ij} g_0$, and hence $m_{ij} = \delta_{ij}$ Then by (8), one has

$$\frac{1}{2\pi\sqrt{-1}} \log q_k = \frac{1}{\langle g_0, \Omega\rangle}\langle g_j, \Omega\rangle = \langle -e^j + u_j e^0, e_0\rangle = u_j,$$

where $\langle g_0, \Omega\rangle = 1$ since Ω is normalized.

4.2. **Hodge Theoretic Mirror Symmetry.** For a pair of Calabi-Yau 3-folds X and $\check{X}$ such that $h^{1,1}(X) = h^{2,1}(\check{X}) = r$, we have two kinds of variation of Hodge structures on Kähler moduli of X, and on complex moduli of X, which we both model as $(\Delta^*)^r$. Furthermore, they both have 0 as maximally unipotent boundary point, and hence the mixed Hodge structure induced by monodromy weight filtration and limiting Hodge filtration at 0 is Hodge-Tate type. We list the corresponding ingredients on both sides in the following table.

	A-model on X	B-model on $\check{X}$
Moduli space:	Kähler moduli space	complex moduli space
Bundle	$\mathcal{H}^{\text{middle}} = \oplus_{p=0}^{3} H^{p,p}(X) \times S$	$\mathcal{H}_{\check{X}} = R^3\pi_*\mathbb{C} \otimes \mathcal{O}_S$
Connection	A-model connection ∇^{middle}	Gauss-Manin connection ∇^{GM}
Polarization	$S(\alpha,\beta) = (-1)^p \int_X \alpha \cup \beta$	$Q(\alpha,\beta) = -\int_{\check{X}} \alpha \cup \beta$
Distinguised Section	$[\text{pt}] \in H^{3,3}(X,\mathbb{Z}) = W_0$	$g_0 \in W_0$
	$1 = [X] = T_0 \in H^0(X,\mathbb{C}) = F^3$	normalized 3-form $\Omega \in \mathcal{F}^3$

As a result, we can now formulate the phenomenon of mirror symmetry into a Hodge-theoretic statement.

Definition 3. Given X and $\check{X}$ are smooth Calabi-Yau 3-folds. $(X, \check{X})$ is a **Hodge-theoretic mirror pair** if the mirror map $M : (\Delta^*)^r \subset \overline{M}(\check{X}) \to (\Delta^*)^r \subset \overline{\mathcal{KM}}(X)$ lifts to a bundle isomorphism between $\mathcal{H}_{\check{X}}$ and $\mathcal{H}_X$. Moreover, this bundle isomorphism preserves the PVHS from ∇^{GM} and ∇^{middle} respectively, and sends the distinguished sections Ω and g_0 of $\mathcal{H}_{\mathcal{X}}$ to 1 and $[pt]$ of $\mathcal{H}^{\text{middle}}$.

Next, we see how above Hodge-theoretic mirror pair leads to the original version of mirror symmetry (cf. sec.1) in terms of correlation functions. First of all, on X, given $a, b, c \in H^2(X, \mathbb{C})$, we have

$$\langle a, b, c\rangle = \int_X a *_s b *_s c,$$

by easy degree counting. Therefore, for $T_i, T_j, T_k \in H^2(X, \mathbb{C})$, we have

(10)

$$\langle T_i, T_j, T_k\rangle = \int_X T_i *_s T_j *_s T_k = \int_X \nabla_{\partial/\partial u_i} \nabla_{\partial/\partial u_j} \nabla_{\partial/\partial u_k} 1 = S(1, \nabla_{\partial/\partial u_i} \nabla_{\partial/\partial u_j} \nabla_{\partial/\partial u_k} 1).$$

On the other hand, on $\check{X}$, we have canonical coordinates $(q_1, \ldots, q_r)$ and we have shown in sec.4.1 that the canonical coordinates are exactly in the image of mirror map. FFrom sec.2.3, we know that under the canonical coordiantes q_i, the normalized Yukawa coupling for $\check{X}$ is given by

$$Y_{ijk} = -\int_X \widetilde{\Omega} \wedge \nabla^{GM}_{\partial/\partial u_i} \nabla^{GM}_{\partial/\partial u_j} \nabla^{GM}_{\partial/\partial u_k} \widetilde{\Omega} = Q(\widetilde{\Omega}, \nabla^{GM}_{\partial/\partial u_i} \nabla^{GM}_{\partial/\partial u_j} \nabla^{GM}_{\partial/\partial u_k} \widetilde{\Omega}),$$

where $\widetilde{\Omega} = \Omega/y_0$ is the normalized 3-form. Since under canonical coordinates q_i, the Gauss-Manin connection is determined by normalized Yukawa coupling with respect to the sections e_i, e^i given in sec.2.3. On the other hand, the A-model connection is determined by the (A-model) three point correlation function $\langle T_i, T_j, T_k\rangle$ with respect to the sections T_i. Therefore, we have

Proposition 10. *X and $\check{X}$ are Hodge-theoretic mirror pair if and only if*

$$Y_{ijk} = \langle T_i, T_j, T_k\rangle$$

That is, the Hodge-theoretic mirror symmetry is equivalent to the mirror symmetry in terms of correlation functions.

Moreover, on $\check{X}$, recall that proposition 1 asserts that there exists a potential Φ^{GM} such that

$$Y_{ijk} = \frac{1}{(2\pi\sqrt{-1})^3} \frac{\partial^3 \Phi^{GM}}{\partial u_i \partial u_j \partial u_k}.$$

Turning to X, we have the Gromov-Witten potential

$$\Phi^{GW} = \frac{(2\pi\sqrt{-1})^3}{6} \int_X \left(\sum_{j=1}^r u_j T_j\right) + \sum_{\beta \neq 0} N_\beta q^\beta,$$

where $q^\beta = \prod_i q_i^{\int_\beta T_i}$ and $q_i = \exp(2\pi\sqrt{-1}u_j)$. Therefore, to establish the equality between correlations function, it suffices to prove

$$\Phi^{GM} = \Phi^{GW}$$

up to a quadratic in u_i. And this is the statement **Mirror theorem** in [12] we will encounter in sec.5.

4.3. **Batyrev's Mirror Construction.** So far, we have formulated a mathematical statement for mirror symmetry for a *mirror pair* of Calabi-Yau 3-folds. However, we have not discussed how to find mirror pair of Calabi-Yau 3-folds. In this subsection, we introduce Batyrev's recipe to construct mirror candidate via toric geometry following [5],[1]. Batyrev's construction of mirror is based on the combinatorial duality between polytopes. Let N be a lattice of rank n, M be its dual lattice, and $\langle\,,\,\rangle$ be the pairing.

Definition 4. A full dimensional lattice polytope $P \subset M_\mathbb{R} \cong \mathbb{R}^n$ is *reflexive* if

(1) The facet representation of P is given by $P = \{m \in M_\mathbb{R} : \langle m, v_F\rangle \leq -1\}$ with all $v_F \in N$.
(2) $\mathrm{int}(P) \cap M = \{0\}$.

It is obvious from the definition that P is reflexive if and only if its dual polytope $P^\vee := \{n \in N_\mathbb{R} : \langle m, n\rangle \leq -1, \forall m \in P\}$ is reflexive. Let X_P be the toric variety constructed from the normal fan of P, the facet normals in the facet representation of P are exactly one-dimensional cone generator of the normal fan of P. Hence, the associated support function ψ_P corresponds to the anti-canonical divisor $-K_{X_P} = \sum_\rho D_\rho$. Therefore, $-K_{X_P}$ is ample divisor, and hence X_P is Gorenstein Fano. Conversely, given any Gorenstein Fano toric variety X, the polytope P_{K_X} associated to the anti-canonical bundle is a reflexive polytope. Hence, P is reflexive if and only if X_P is Gorenstein Fano toric variety. By the same token, we can obtain another Gorenstein Fano variety $X_{P^\vee}$.

Remark 4. Although X_P and $X_{P^\vee}$ seem to be similar at this point of view, they in fact behave very differently. For instance, if P is the cube in $\mathbb{R}^3$ ceneted at the orgin with vertices $(\pm1, \pm1, \pm1)$, the corresponding toric variety $X_P = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is smooth. However, the dual polytope $P^\vee$ is octahedron with vertices $\pm e_1, \pm e_2, \pm e_3$, and the corresponding toric variety $X_{P^\vee}$ is not even an orbifold.

From the above remark, we can see that the Gorenstein Fano toric varieties obtained from reflexive polytope in general very singular. Therefore, Batyrev used the following toric "partial resolution", called maximal projective crepant partial (MPCP) resolution in [1]. Given a reflexive polytope P, the refinement Σ of the normal fan of P in $N_\mathbb{R}$ such that

(1) $\Sigma(1) = P^\vee \cap N - \{0\}$
(2) Σ is simplicial.

The corresponding projective toric morphism $f : X_\Sigma \to X_P$ is the desired MPCP resolution.

Proposition 11. *The MPCP resolution $f : X_\Sigma \to X_P$ has the following properties:*

(1) *f is crepant and $-K_{X_\Sigma}$ is semi-ample invertible sheaves.*
(2) *X_Σ is a toric orbifold with at worst terminal singularities.*

Proof. (1) $P^\vee$ is reflexive as well and $\Sigma(1) = P^\vee \cap N - \{0\}$, the facet normal m of $P^\vee$ give the Cartier data for $-K_{X_\Sigma}$. Hence, K_{X_Σ} is Cartier as well. Next, since $-K_{X_P}$ is ample, $f^*(-K_{X_P})$ is semi-ample. Hence, if we know that f is crepant, then $-K_{X_\Sigma}$ is semi-ample. Since $K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho$, the discrepancy is given by:

$$K_{X_\Sigma} - f^*(K_{X_P}) = -\sum_{\rho\in\Sigma(1)-\mathrm{Ver}(P^\vee)} (1 + \langle v_\rho, m\rangle)D_\rho,$$

where v_ρ is the primitive generator of the ray ρ. Since $P^\vee$ reflexive, all v_ρ lies on the facet of $P^\vee$. Hence, $\langle v_\rho, m\rangle = -1$. Therefore, f is crepant.

(2) By construction of MPCP resolution, Σ is simplicial fan. Thus, X_Σ is an orbifold readily. Next, we prove that X_Σ has at worst terminal singularities. Since $P^\vee$ is the intersection of integral half-spaces $\{v \in N_\mathbb{R} : \langle v, m\rangle \le -1\}$, where m ranges over all vertex of P. Then the condition $\Sigma(1) = P^\vee \cap N - \{0\}$ and $\mathrm{int}(P^\vee) = \{0\}$ imply that on each maximal cone σ of Σ,
   (a) all 1-dimension cone generator $\rho_1, \dots, \rho_s$ of σ are contained in the integral affine hyperplane $(m(n) = -1)$
   (b) $N \cap \sigma \cap (m(n) < -1) = \{0, \rho_1 \dots, \rho_s\}$.

   Therefore, by characterization of terminal singularity given in [15], we see that X_Σ has at worst terminal singularity.

□

Since $-K_{X_\sigma}$ semiample and general hypersurfaces inherits the singularities of the ambient space, general hypersurface $V \in |-K_{X_\Sigma}|$, V is a minimal Calabi-Yau orbifold. Also, since f is crepant, such Calabi-Yau hypersurface can also be regarded as proper transform of general anti-canonical hypersurface $\bar{V} \in |-K_{X_P}|$.

Remark 5. In general, for $\dim P \ge 3$, one cannot expect to have toric crepant resolution for X_P. However, there is a simple fact that the singular locus of a toric orbifold with at worst terminal singularities has codimension greater or equal than 4. ([17], lemma 4.16,17) Hence, for $\dim P = 4$, under MPCP resolution, X_Σ has only isolated singularities. Therefore, the general anti-canonical hypersurface is a smooth Calabi-Yau 3-fold. Alternatively, one can prove that the singular locus of a Gorenstein orbifold with at worst terminal singularities has codimension ≥ 4 ([5], Prop. A.2.2). Thus, V is again smooth.

Now, for a 4-dimension reflexive polytope P and its dual polytope $P^\vee$, we can repeat the same construction on $P^\vee$ to obtain a smooth Calabi-Yau 3-fold $V^\vee \in |-K_{X_{\Sigma^\vee}}|$, where $\Sigma^\vee$ is the refinement of the normal fan of $P^\vee$. We call $(V, V^\vee)$ the *Batyrev's mirror pair.* The first evidence of being mirror pair is the hodge number duality:

$$h^{2,1}(V^\vee) = h^{1,1}(V); h^{1,1}(V^\vee) = h^{2,1}(V).$$

For this, Batyrev proved the following combinatorics formula for the hodge numbers of V([1], Thm. 4.3.7, 4.4.2):

$$h^{1,1}(V) = l(P^\vee) - 5 - \sum_{\Sigma^\vee} l^*(\Sigma^\vee) + \sum_{\Theta^\vee} l^*(\Theta^\vee) l^*(\widehat{\Theta}^\vee) \tag{11}$$

$$h^{2,1}(V) = l(P) - 5 - \sum_{\Sigma} l^*(\Sigma) + \sum_{\Theta} l^*(\Theta) l^*(\widehat{\Theta}), \tag{12}$$

where $\Sigma^\vee$ ranges over all facet of $P^\vee$, $\Theta^\vee$ ranges over all codimension two face of $P^\vee$, $\widehat{\Theta}^\vee$ are the faces of P dual to $\Theta^\vee$. Also, $l(\,)$ and $l^*(\,)$ mean the number of lattice point on the polytopes (faces) and on their interior resepctively. From the symmetry of above formulae, the hodge number duality follows directly. Note that for each integral points in P corresponds to a global section for anti-canonical bundle. Since scaling the section gives the same hypersurface, $l(P) - 1$ gives the dimension of parameter space of polynomial deformation of $\overline{V}$. Also, we need to cut out the automorphism group coming from X_P, which is of dimension $\dim P + \sum_\Sigma l^*(\Sigma)$ by [3], Proposition 3.6.2. This explains the first three terms of the formula of $h^{2,1}(V)$. For the last term, this is the complex deformation of V that are not hypersurdaces.

Remark 6. In fact, in [1], Batyrev proved above formulae in general dimension. However, since in general dimension, V is just an orbifold, we must use Deligne-Hodge number instead. Batyrev investigated the intersections of the hypersurface with each torus orbit and uses results of Danilov and Khovanskii's result [6] on calculating Deligne-Hodge number for hypersurfaces in torus. We refer the details to original article [6], [1] as well as the survey [17], chapter 4.

5. Classic Example: Quintic Threefolds

In this section, we let X to be smooth quintic hypersurface of $\mathbb{P}^4$.

5.1. Basic Properties on Quintic Threefolds and its Mirror.

In this section, we calculate the Hodge numbers of quintic 3-folds and give the construction of quintic mirror by Batyrev's recipe. First, we calculate the Hodge number of X. By Lefschetz hyperplane theorem, $H^2(X, \mathbb{C}) \cong H^2(\mathbb{P}^4, \mathbb{C}) \cong \mathbb{C}$. Since X is compact Kähler, $h^2(X, \mathbb{C}) = h^{1,1}(X) = 1$. To calculate $h^{2,1}(X)$, we first use Gauss-Bonnet to calculate the topological Euler characteristic of X. From the exact sequence,

$$0 \to TX \to T\mathbb{P}^4|_X \to N_{X/\mathbb{P}^4} \to 0,$$

we have

$$c(T\mathbb{P}^4|_X) = c(TX)c(N_{X/\mathbb{P}^4}) = c(TX)c(O_X(5)) = c(TX)(1 + 5h),$$

where h is the first Chern class of $c_1(O_{\mathbb{P}^4}(1))|_X$. Also, from Euler sequence

$$0 \to O_{\mathbb{P}}^4 \to O_{\mathbb{P}^4}(1)^{\oplus 5} \to T\mathbb{P}^4 \to 0,$$

$$c(TX) = \frac{(1+h)^5}{1+5h} = (1 + 5h + 10h^2 + 10h^3 + \cdots)(1 - 5h + 25h^2 - 125h^3 + \cdots).$$

Hence,

$$\chi_{top}(X) = \int_X c(TX) = \int_X -40h^3 = -200.$$

Since $h^0(X) = h^2(X) = h^4(X) = h^6(X) = 1$, $h^1(X) = h^5(X) = 0$, $\chi_{top}(X) = 4 - h^3(X) = -200$. Thus, we have $h^{2,1}(X) = 101$. The quintic 3-fold has the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Now, for mirror manifold $\check{X}$ of quintic 3-fold X, it should satisfies $h^{2,1}(\check{X}) = 1.$. The construction of quintic mirror was originated given in [9] based on physical argument. Here, we use Batyrev's recipe introduced in sec.4.3 to construct the quintic mirror. First of all, $\mathbb{P}^4$ has the fan generated by the subsets of $v_0 = (-1,-1,-1,-1), v_i = e_i$, for $i = 1, \dots, 4$. Let D_i be the torus invariant Weil divisor corresponding to the ray generated by v_i. Then the polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^4$ corresponding to the anti-canonical divisor $-K_{\mathbb{P}^4} = D_0 + \cdots + D_4$ has vertices:

$$\begin{aligned} v_0 &= (-1,-1,-1,-1) \\ v_1 &= (4,-1,-1,-1) \\ v_2 &= (-1,4,-1,-1) \\ v_3 &= (-1,-1,4,-1) \\ v_4 &= (-1,-1,-1,4) \end{aligned} \tag{13}$$

Then the toric variety $X_{P^\vee}$ is determined by the normal fan of $P^\vee$ in $M_{\mathbb{R}}$, which is just the cone over the faces of P. Hence, the fan $\Sigma^\vee$ associated to $X_{P^\vee}$ has cone generators (13). They generates a lattice $M_1 \subset M$. The index of the lattice can be computed the volume of the polytope P, which is 125. The quotient

$$G := M/M_1 = \{(a_0, a_1, a_2, a_3, a_4) \in (\mathbb{Z}_5)^5 : \sum_{i=0}^{4} a_i \equiv 0 \mod 5\}/\mathbb{Z}_5,$$

where $\mathbb{Z}_5 \subset (\mathbb{Z}_5)^5$ as the diagonal. Thus, $X_{P^\vee} = \mathbb{P}^4/G$. From the exact sequence

$$0 \to M_1 \to Z^{\Sigma^\vee(1)} \to Cl(X_P^\vee) \to 0,$$

we can get a new one

$$0 \to M = \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^5 \to Cl(\mathbb{P}^4) \oplus G \to 0.$$

The first map is the matrix given by (13) and the second one is

$$(a_0, a_1, a_2, a_3, a_4) \mapsto [a_0, a_1, a_2, a_3, a_4] := (\sum_{i=0}^{4} a_i, (-a_1-a_2-a_3-a_4, a_1, a_2, a_3, a_4) \in \mathbb{Z} \oplus G,$$

where we identify $Cl(X_{P^\vee})$ by $\mathbb{Z}\oplus G$. Since $[-K_{X_{P^\vee}}]=[1,1,1,1,1]=(5,0)\in\mathbb{Z}\oplus G$, in terms of the homogenious coordinate $x_0,\dots,x_4$ on $\mathbb{P}^4$ yet graded by $\mathbb{Z}\oplus G$, the monomials having degree $(5,0)$ are given by x_i^5 and $x_0x_1x_2x_3x_4$ (Note that $5e_i=(1,1,1,1,1)=0$ in G). Thus, we obtain a family of hypersurfaces $\overline{V}^\vee\subset X_{P^\vee}$ defined by the quotient of the hypersurfaces

$$(a_0x_0^5+a_1x_1^5+a_2x_2^5+a_3x_3^5+a_4x_4^5+a_5x_0x_1x_2x_3x_5x=0)\subset\mathbb{P}^4$$

by G. Then we obtain the quintic mirror family $\check{X}=V^\vee$ by taking the proper transform of $\overline{V}^\vee$ under MPCP resolution.

On the other hand, since $P^\vee$ has vertices $(-1,-1,-1,-1),e_1,e_2,e_3,e_4$, the only lattices points of $P^\vee$ are just these five along with the origin. Therefore, there are no lattice points on the faces of $P^\vee$. By Batyrev's betti number formula, we have:

$$h^{2,1}(\check{X})=5-5+0+0=1.$$

Moreover, by the discussion in the end of sec.4.3, we see that $\check{X}$ has only polynomial deformation. Therefore, we consider the 1-parameter family which are the G-invariant hypersurfaces in $\mathbb{P}^4$

$$V_\psi^\vee:=(x_0^5+x_1^5+x_2^5+x_3^5+x_4^5+\psi^5x_0x_1x_2x_3x_5x=0), \tag{14}$$

where we identify the action of G on $\mathbb{P}^4$ by

$$(a_0,a_1,a_2,a_3,a_4)\cdot(x_0,\dots,x_5)=(\mu^{a_0}x_0,\dots,\mu^{a_4}x_4).$$

(μ is the primitive fifth root of unity) From (14), we see that $(x_0,\dots,x_4)\mapsto(\mu^{-1},x_0,\dots,x_5)$ induces an isomorphism $V_\psi^\vee\cong V_{\mu\psi}^\vee$. Thus, ψ^5 is well-defined on complex moduli space of $\check{X}$. $z:=\psi^{-5}$ is a local coordinate on the complex moduli of $\check{X}$. At $z=0$, the family of hypersurfaces degenerate to $x_0\cdots x_5=0$, which is singular. In 5.3, we will show that $z=0$ is a LCSL point.

5.2. **Quantum Cohomology on Quintic 3-fold.** In this section, we turn to the A-model on smooth quintic 3-fold. First, since $h^{1,1}(X)=1$ and $H^2(X)=H^{1,1}(X)$ is generated by the hyperplane class H. We have the single variable u corresponding to H. Also, the homology $H^2(X,\mathbb{Z})$ is generated by a line $l\subset X$, we denote Gromov-Witten invariants $\langle,\rangle_{0.n,d[l]}$ by $\langle\,\rangle_{0,n,d}$, and $N_{d[l]}:=\langle\,\rangle_{0,0,d}$ by N_d. From sec.3.2, we know that Gromov-Witten invariants on X are determined by N_d and intersection numbers. Now, the Gromov-Witten potential of quintic 3-fold restricting to $H^2(X,\mathbb{C})$ is given by

$$\begin{aligned}\Phi^{GW}(t)&=\sum_{n=0}^\infty\sum_{d=0}^\infty\frac{1}{n!}\langle(tH)^n\rangle_{0,n,d}\\&=\frac{t^3}{6}\int_X H^3+\sum_{d\geq1}\sum_n\frac{(dt)^n}{n!}N_d=\frac{5t^3}{6}+\sum_{d\geq1}N_de^{dt}\\&=\frac{5t^3}{6}+\sum_{d\geq1}N_dq^d,\end{aligned}\tag{15}$$

where $q = e^t$. Also, following Kontsevich's proposol [11], one can reduce the problem of computing

$$N_d = \langle\,\rangle_{0,0,d} = \deg([\overline{M}_{0,0}(X,d)]^{\mathrm{vir}})$$

to Gromov-Witten invariants of $\mathbb{P}^4$ as follows. From the imbedding $X \subset \mathbb{P}^4$, this induces an embedding $i : \overline{M}_{0,0}(X,d) \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^4,d)$. Then from the construction of virtual fundamental class of $\overline{M}_{0,0}(X,d)$ (cf. [3], Example 7.1.5.1), one can show that

$$i_*[\overline{M}_{0,0}(X,d)] = c_{5d+1}(U_d) \cap [\overline{M}_{0,0}(\mathbb{P}^4,d)].$$

Here, U_d is the vector bundle on $\overline{M}_{0,0}(\mathbb{P}^4,d)$ whose fiber over a genus 0 stable map f is given by $H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))$. Here, $5d+1$ is the dimension of the cohomology $H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))$. Since

$$\mathrm{v.}\dim_{\mathbb{C}} \overline{M}_{0,0}(\mathbb{P}^4,d) = \int_{d[l]} 5H + 1 = 5d+1,$$

we know that$c_{5d+1}(U_d)$ is a top cohomology class. Hence,

$$N_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4,d)} c_{5d+1}(U_d).$$

5.3. **Picard-Fuchs Equation and Yukawa Coupling on Quintic Mirror.** In this section, we investigate the B-model on quintic mirror family. First, we compute the Picard-Fuchs equation of mirror family. We will do this by Griffths-Dwork method following [10]. Let

$$\Omega_0 := \sum_{j=0}^{4} (-1)^j x_j dx_0 \wedge \cdots \wedge \hat{dx_j} \wedge dx_4$$

be the holomorphic volume form on $\mathbb{P}^4$. Since the mirror family $\check{X}_\psi$ is the quotient of the hypersurfaces

$$f_\psi := x_0^5 + \cdots x_4^5 + \psi x_1 \cdots x_5 = 0$$

by G. We apply the Griffths-Dowrk method for the G-invariant forms $P\Omega_0/f^k$ on the complement of the hypersurface in $\mathbb{P}^4$ to represent cohomology classes on $\check{X}$. For $k = 1,2,3,4$, let

$$\omega_k = \frac{(-1)^{j-1}\psi^j (\prod_i x_i)^{j-1}\Omega_0}{f_\psi^j}.$$

They are G-invariant and invariant under the change $x_0 \mapsto \mu^{-1}x_0$, $\psi \mapsto \mu\psi$. Since $\mathrm{Res}(\omega_k) \in F^{4-j}H^3(\check{x},\mathbb{C})$. Hence, the Picard-Fuchs equation for ω_1 can be written in terms of moduli coordinate $z = \psi^{-5}$. Let $\delta = zd/dz = \frac{-1}{5}\psi d/d\psi$, one compute directly that

$$\delta\omega_k = \frac{-1}{5}\psi\frac{d\omega_k}{d\psi} = \frac{-k}{5}(\omega_k + \omega_{k+1}.$$

Therefore, we have:

$$\begin{bmatrix} \omega_1 \\ \delta\omega_1 \\ \delta^2\omega_1 \\ \delta^3\omega_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/5 & -1/5 & 0 & 0 \\ 1/25 & 3/25 & 2/25 & 0 \\ -1/125 & -1/125 & -12/125 & -6/125 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

On the other hand, using pole order reduction and calculation by Gröbner basis technique, we can find a relation

$$\delta^4\omega_1 = \begin{bmatrix} 5z/1+5^5z & 75z/1+5^5z & 250z/1+1+5^5z & 300z/1+5^5z \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

Together we obtain:

$$\delta^4\omega + \frac{2\cdot 5^5 z}{1+5^5z}\delta^3\omega + \frac{7\cdot 5^4 z}{1+5^5z}\delta^2\omega + \frac{2\cdot 5^4 z}{1+5^5z}\delta + \frac{24\cdot 5}{1+5^5z}\omega = 0$$

Therefore, we have the Picard-Fuchs equation of quintic mirror.

$$\delta^4 y + \frac{2\cdot 5^5 z}{1+5^5z}\delta^3 y + \frac{7\cdot 5^4 z}{1+5^5z}\delta^2 y + \frac{2\cdot 5^4 z}{1+5^5z}\delta + \frac{24\cdot 5}{1+5^5z}y = 0 \tag{16}$$

One can see that (16) has $z=0$ as regular singular point. Moreover, the at $z=0$, the indicial equation of (16) is $\lambda^4 = 0$. Following [3] Proposition 5.2.1, we can show that monodromy operator about $z=0$ is of maximal unipotency index. Hence, $z=0$ is a LCSL point. Also, notice that the equation can be written as the following factorization

$$L = \delta^4 + 5z(5\delta+1)(5\delta+2)(5\delta+3)(5\delta+4) = 0 \tag{17}$$

To obtain a solution of Picard-Fuchs equation, we consider the period on the affine part $x_0 = 1$,

$$\frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{\psi dx_1 dx_2 dx_3 dx_4}{1+x_1^5+x_2^5+x_3^5+x_4^5+\psi x_1x_2x_3x_4x_5},$$

where $\gamma = \{[1 : x_1 : x_2 : x_3 : x_4] : |x_i| = \epsilon\}$. Now, we expand the period

$$\begin{aligned}
&\frac{1}{(2\pi\sqrt{-1})^4}\int_\gamma \frac{\psi dx_1 dx_2 dx_3 dx_4}{1+x_1^5+x_2^5+x_3^5+x_4^5+\psi x_1x_2x_3x_4x_5} \\
=&\frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{dx_1 dx_2 dx_3 dx_4}{\psi^{-1}(1+x_1^5+x_2^5+x_3^5+x_4^5)+x_1x_2x_3x_4} \\
=&\frac{1}{(2\pi\sqrt{-1})^4}\int_\gamma \frac{dx_1 dx_2 dx_3 dx_4}{x_1x_2x_3x_4}\frac{1}{1+(1+x_1^5+x_2^5+x_3^5+x_4^5)/\psi(x_1x_2x_3x_4)} \\
=&\frac{1}{(2\pi\sqrt{-1})^4}\sum_{n=0}^{\infty}\int_\gamma \frac{dx_1 dx_2 dx_3 dx_4}{x_1x_2x_3x_4}\frac{(-1)^n(1+x_1^5+x_2^5+x_3^5+x_4^5)^n}{\psi^n(x_1x_2x_3x_4)^n}
\end{aligned}$$

Taking resiude, the only terms which will contribute is the constant term in the second facotr of the integrad. Therefore, we have

$$\frac{1}{(2\pi\sqrt{-1})^4}\int_\gamma \frac{\psi dx_1 dx_2 dx_3 dx_4}{1+x_1^5+x_2^5+x_3^5+x_4^5+\psi x_1x_2x_3x_4x_5} = \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5(-\psi)^5 n} = \sum_{n=0}^{\infty}\frac{(5n)!(-z)^n}{(n!)^5}.$$

Note that above power series converge near $z=0$. Since $z=0$ is a LCSL point, $y_0(z) = \sum_{n=0}^{\infty}\frac{(5n)!(-z)^n}{(n!)^5}$ is the unique holomorphic solution y_0 for (16).

Now, we can use Frobenius method ([4], Chap. 4.8) to obtain other three solutions as following. We first change of variable $x = -z$, and $\delta = x\frac{d}{dx}$, then (16) becomes

$$L = \delta^4 - 5x(5\delta+1)(5\delta+2)(5\delta+3)(5\delta+4)$$

Then consider the auxillary function

$$\tilde{y}(x,\lambda) = \sum_{n=0}^{\infty} a_n(\lambda) x^{n+\lambda}.$$

We require $\tilde{y}(x,\lambda)$ to satisfy

$$L(\tilde{y}(x,\lambda)) = \lambda^4 x^\lambda, \tag{18}$$

where λ^4 is the indicial equation of (16) at $x = 0$. We then have the recursion relation

$$(\lambda+n)^4 a_n(\lambda) = 5(5\lambda+5n-1)(5\lambda+5n-2)(5\lambda+5n-3)(5\lambda+5n-4)a_{n-1}(\lambda).$$

Choosing the solution with initial condition $a_0 = 1$, we then have:

$$\tilde{y}(x,\lambda) = \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{5n}(5\lambda+m)}{\prod_{m=1}^{n}(\lambda+m)^5} x^{n+\lambda}.$$

Then from (18), for $k = 0, 1, 2, 3$,

$$y_k(x) = \frac{1}{k}\frac{\partial^k \tilde{y}(x,\lambda)}{\partial \lambda^k}|_{\lambda=0}$$

give a basis of solution near $x = 0$. In particular, for $k = 1$,

$$\left.\frac{\partial \tilde{y}(x,\lambda)}{\partial \lambda}\right|_{\lambda=0} = \sum_{n=0}^{\infty} \left.\frac{\partial a_n(\lambda)}{\partial \lambda}\right|_{\lambda=0} x^n + \log(x) y_0(x)$$

Then

$$\begin{aligned}
\left.\frac{\partial a_n(\lambda)}{\partial \lambda}\right|_{\lambda=0} &= \frac{\partial}{\partial \lambda}\left(\frac{\Gamma(\lambda+1)^5\Gamma(5n+5\lambda+1)}{\Gamma(5\lambda+1)\Gamma(n+\lambda+1)^5}\right)\Bigg|_{\lambda=0} \\
&= 5\frac{((5n)!\Gamma'(1)+\Gamma'(5n+1)}{(n!)^5} - 5(5n)!\frac{(n!)^4\Gamma'(n+1)+(n!)^5\Gamma'(1)}{(n!)^{10}} \\
&= 5\frac{\Gamma'(5n+1)}{(n!)^5} - 5\frac{(5n)!\Gamma'(n+1)}{(n!)^6}
\end{aligned}$$

For derivative of Γ-function evaluating at positive integers, we have the formula

$$\Gamma'(n+1) = \Gamma(n+1)\left(-\gamma + \sum_{j=1}^{n}\frac{1}{j}\right) = n!\left(-\gamma + \sum_{j=1}^{n}\frac{1}{j}\right),$$

where γ is the Euler–Mascheroni constant. Hence, we have

$$\left.\frac{\partial a_n(\lambda)}{\partial \lambda}\right|_{\lambda=0} = \frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right).$$

This gives

$$y_1(z) = y_0(z)\log(-z) + 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)(-z)^n$$

, which is the logarithmic solution of 16. Therefore, from (9), the mirror map is given by

$$q = -z\exp\left(\frac{5}{y_0(z)}\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\sum_{j=n+1}^{5n}\frac{1}{j}(-z)^n\right).$$

In section2.3, we have shown that the normalized Yukawa coupling is the third derivative of some potential function Φ^{GM}. We now establish the formula of potential function for quintic mirror.

Proposition 12. *The potential function for the quintic mirror family is*

$$\Phi^{GM} = \frac{5}{2}\left(\frac{y_1y_2}{y_0^2} - \frac{y_3}{y_0}\right),$$

regarded as a function of $t = \log q$ *using the mirror map* $q = \exp(y_1/y_0)$

Proof. In Prop.1, we use the coordinate u such that $q = \exp(2\pi\sqrt{-1}u)$. Now, we let $t = 2\pi\sqrt{-1}u$. Then, we need to show that

$$\frac{d^3}{dt^3}\frac{5}{2}\left(\frac{y_1y_2}{y_0^2} - \frac{y_3}{y_0}\right) = Y,$$

where Y is the Yukawa coupling. Recall in sec.2.3, the mirror map q is the canonical coordinate on Δ^*, and there exists local section e_0, e_1, e^1, e^0, where e_0 is the normalized 3-form Ω, such that

$$\begin{aligned}\nabla_\delta e_0 &= e_1,\\ \nabla_\delta e_1 &= Ye^1,\\ \nabla_\delta e^1 &= e^0,\\ \nabla_\delta e^0 &= 0.\end{aligned}$$

Here, $\delta = 2\pi\sqrt{-1}qd/dq$ and Y is the normalized Yukawa coupling. Hence, in q coordinate, the Picard-Fuchs equation takes a simpler form

$$\nabla_\delta^2\left(\frac{\nabla_\delta^2 e_0}{Y}\right) = 0.$$

Then the Picard-Fuchs equation becomes

$$\frac{d^2}{dt^2}\left(\frac{d^2y/dt^2}{Y}\right) = 0.$$

Then the equation has solution $f_0 = 1, f_1 = t, f_2" = Y, f_3" = Yt$. On the other hand, y_i/y_0 are also solutions for normalized equation, for $i = 0, 1, 2, 3$. In [3] Example 5.6.4.1, we have the explicit formula for normalized Yukawa coupling

$$Y = \frac{5}{(1+5^5z)y_0^2}\left(\frac{q}{z}\frac{dz}{dz}\right)^3.$$

Thus, we can show that $f_2" = \frac{5d^2}{dt^2}(y_2/y_0) = Y$ and $f_3" = \frac{5d^2}{dt^2}(y_3/y_0) = Yt$ by comparing their expansion in q (cf. [5], 2.6.2). Thus, we have

$$\frac{d^2}{dt^2}\frac{5}{2}(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}) = \frac{d^2}{dt^2}(t\frac{y_2}{y_0} - \frac{y_3}{y_0}) = 5\frac{d}{dt}\frac{y_2}{y_0}.$$

□

We conclude by stating the **Mirror theorem** proved in [12].

Theorem 2. *Under $T(t) = \frac{y_1}{y_0}$, we have*

$$\Phi^{GW}(T(t)) = \frac{5T^3}{6} + \sum_{d>0} N_d e^{dT} = \Phi^{GM}(t) = \frac{5}{2}(\frac{y_1 y_2}{y_0^2} - \frac{y_3}{y_0}).$$

References

[1] Victor Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, J. Alg. Geom. **3** (1993), 493–545.

[2] Philip Candelas, C Xenia, Paul S Green, and Linda Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nuclear Physics B **359** (1991), no. 1, 21–74.

[3] Eduardo Cattani and Aroldo Kaplan, *Polarized mixed Hodge Structures and the Local Monodromy of a Variation of Hodge Structure*, Inventiones mathematicae **67** (1982), no. 1, 101–115.

[4] Earl A Coddington and Norman Levinson, *Theory of Ordinary Differential Equations*, Tata McGraw-Hill Education, 1955.

[5] David A Cox and Sheldon Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical Surveys and Monographs, vol. 68, American Mathematical Soc., 1999.

[6] V. Danilov and Khovanskii, *Newton Polyhedra and an Algorithm for Computing Hodge–Deligne Numbers*, Mathematics of the USSR-Izvestiya **29** (1987), no. 2, 279.

[7] Pierre Deligne, *Local behavior of Hodge structures at infinity*, Mirror Symmetry II, 1997, pp. 683–699.

[8] Alexander B Givental, *Equivariant Gromov-Witten Invariants*, International Mathematics Research Notices **1996** (1996), no. 13, 613–663.

[9] Brian Greene and Ronen Plesser, *Duality in Calabi-Yau Moduli Space*, Nuclear Physics B **338** (1990), no. 1, 15–37.

[10] Philip A Griffiths, *On the Periods of Certain Rational Integrals: I*, Annals of Mathematics (1969), 460–495.

[11] Maxim Kontsevich, *Enumeration of Rational Curves via Torus Actions*, The Moduli Space of Curves, 1995, pp. 335–368.

[12] Bong H Lian, Kefeng Liu, and Shing-Tung Yau, *Mirror Principle I*, Asian Journal of Mathematics **1** (1997), no. 4, 729–763.

[13] Yuri Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, American Mathematical Soc., 1999.

[14] David Morrison, *Compactifications of Moduli Spaces Inspired by Mirror Symmetry*, Astérisque **218** (1993), 243–271.

[15] Miles Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, 1983, pp. 395–418.

[16] Wilfried Schmid, *Variation of Hodge structure: the Singularities of the Period Mapping*, Inventiones mathematicae **22** (1973), no. 3, 211–319.

[17] Claire Voisin, *MIrror Symmetry*, SMF/AMS Texts and Monographs, American Mathematical Soc., 1999.

[18] ———, *Hodge Theory and Complex Algebraic Geometry Vol.I*, Cambridge University Press, 2002.