LOCALIZATIONS ON $\mathbb{P}^2$

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ABSTRACT. In this note, we record the current status on using localizations technique to compute the Gromov–Witten invariants

$$\int_{\overline{M}_{0,d+1}(\mathbb{P}^2,d)} \mathrm{e}(\mathcal{V})\mathrm{ev}_1^*h^2\cdots\mathrm{ev}_{d+1}^*h^2.$$

Here $\mathcal{V}\to\overline{M}_{0,d+1}(\mathbb{P}^2,d)$ is the induced bundle of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^2$ and h is the hyperplane class in $\mathbb{P}^2$.

1. Introduction

In [7], Manin used localization technique proposed in [4] to prove genus zero multiple cover formula for $\mathbb{P}^1$, which is

$$\int_{\overline{M}_{0,0}(\mathbb{P}^1,d)} e(\mathcal{V}_d)=1/d^3,$$

where $\mathcal{V}_d\to\overline{M}_{0,0}(\mathbb{P}^1,d)$ is the induced bundle of $O_{\mathbb{P}^1}(-1)^{\oplus 2}$ over $\mathbb{P}^1$. Higher genus case on $\mathbb{P}^1$ was settled by Graber, Faber, and Pandharipande [3],[2]. While for higher dimensional case, Lee, Lin, and Wang [5] proved

$$\int_{[\overline{M}_{0,n}(\mathbb{P}^r,d)]^{\mathrm{vir}}} e(\mathcal{V}_d)\prod_{i=1}^{n}\mathrm{ev}_i^*(h^{l_i})=(-1)^{(r+1)(d-1)}u(l_1,\dots,l_n)d^{n-3},$$

where $u(l_1,\dots,l_n)$ is a universal constant independent of d, $\mathcal{V}_d$ is the induced bundle on $\overline{M}_{0,n}(\mathbb{P}^r,d)$ by $\mathcal{O}(-1)^{\oplus(r+1)}$ over $\mathbb{P}^r$, $l_1,\dots,l_n$ are positive integers such that $\sum_i l_i=2r+n-2$, and h is the hyperplane class on $\mathbb{P}^r$. Their proof is more involved. Later, Tsai [8] gave a new proof by direct localizations, and he also determined the universal constant above. Recently, Lee, Lin, and Wang [6] tackle the case when r' in the bundle $\mathcal{O}(-1)^{\oplus(r'+1)}$ is less than the base r. The simplest case is when $(r,r')=(2,1)$, and they proved the following theorem using more advanced tool.

Theorem 1.

$$\int_{[\overline{M}_{0,d+1}(\mathbb{P}^2,d)]} c_{2d-2}(U_d)\cup\prod_{i=1}^{d+1}\mathrm{ev}_i^*h^2=d^{d-2},$$

where U_d be the induced bundle of $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^2$ on $\overline{M}_{0,d+1}(\mathbb{P}^2,d)$, and h is the hyperplane class.

This note records the current status on proving above theorem by direct localization method. More specifically, we verified the theorem by localization when $d=2,3$.

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2. Set Up:Localizations on $\overline{M}_{g,n}(\mathbb{P}^r, d)$

2.1. **Localizations on $\mathbb{P}^r$.** Here, following [4], we recall the combinatorial scheme of localization on moduli spaces of stable maps. Our main reference is [1], chapter 9. Consider a torus $T = (\mathbb{C}^*)^{r+1}$ action on $\mathbb{P}^r$ in the standard way. This induces an T-action on the moduli space of stable maps to $\mathbb{P}^r$ and on its universal family:

$$\overline{U}_{g,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d). \tag{1}$$

A stable map is denoted by $(f, C, x_1, \dots, x_n)$ or simply by f if the context is clear.

Let $\{p_0, \dots, p_r\}$ be the set of T-fixed points on $\mathbb{P}^r$. Now for a T-fixed point f of $\overline{M}_{g,n}(\mathbb{P}^r, d)$, it satisfies:

1. $f(x_i) \in \{p_0, \dots, p_r\}$
2. For each irreducible component C_i of C.
   - (a) If $\dim f(C_i) = 0$, i.e. C_i is a contracted component, then $f(C_i) \in \{p_0, \dots, p_r\}$.
   - (b) If $\dim f(C_i) = 1$, then $f(C_i)$ is some coordinate line ℓ_{ij} connecting p_i and p_j. Hence, the morphism restricting to the component $f|_{C_i} : C_i \to \mathbb{P}^1$ is a degree d_i map. Being T-invariant means the ramification locus must lie over $0, \infty \in \mathbb{P}^1$.

   In the second case, Riemann–Hurwitz Formula implies that $2g(C_i) - 2 < 0$. Hence, C_i can only be rational.

For each T-fixed morphism f, we can associate a graph Γ with various labels as follows:

1. Vertices: For $v \in V(\Gamma)$, we assign to a connected components C_v of $f^{-1}(\{p_0, \dots, p_r\})$. Hence, C_v is either a point or unions of contracted components. For each vertex v, we associate several additional labels:
   - (a) The number $i(v)$ is defined by: $f(C_v) = p_{i(v)}$.
   - (b) If $\dim C_v = 1$, then define its genus $g(v) := g_a(C_v)$, the arithmetic genus of C_v. We define the genus of the graph Γ by:

     $$g(\Gamma) := b_1(\Gamma) + \sum_{v \in V(\Gamma)} g(v),$$

     where b_1 is the first Betti number of the graph.
   - (c) For each vertex v, denote $S_v := \{i \in [n] : x_i \in C_v\}$ and $N_v := |S_v|$, and we attach each vertex v a tail with number i if the i-th marked point x_i lies in

C_v. Obviously, we have

$$\{1,\dots,n\} = \coprod_{v\in V(\Gamma)} S_v.$$

(2) Edges $E(\Gamma)$: For $e \in E(\Gamma)$, we associate to a non-contracted components C_e of C. Moreover, for each edge C_e, we define the labeling d_e to be the degree of $f|_{C_e}$. Define the degree of graph $d(\Gamma) := \sum_{e\in E(\Gamma)} d_e$.

The valency of a vertex v, denoted by $\mathrm{val}(v)$, is defined to be the number of edges incident to v, and $n(v) := \mathrm{val}(v) + N_v$ is the sum of number of edges and number of tails incident to the vertex v.

Given a triple of non-negative integer (g, n, d), we define the set $S_{g,n,d}$ to be the set of graphs Γ with the above structures such that $g(\Gamma) = g, d(\Gamma) = d$, and $\sum_{v\in V(\Gamma)} N_v = n$. Then given such a graph $\Gamma \in S_{g,n,d}$, we define:

$$M_\Gamma = \prod_{v\in V(\Gamma)} \overline{M}_{g(v),n(v)},$$

where $\overline{M}_{g(v),n(v)}$ is the Deligne–Mumford space of $n(v)$-pointed, genus $g(v)$ stable curves. Denote $U_\Gamma \longrightarrow M_\Gamma$ to be the universal family.

We define a morphism $U_\Gamma \to \mathbb{P}^r$ by: given $(C_v)_{v\in V(\Gamma)} \in U_\Gamma$, we connect C_v and $C_{v'}$ by $C_e \simeq \mathbb{P}^1$ if an edge e connects vertices v and v', and define

$$f : C = \bigcup_{i\in V(\Gamma)\cup E(\Gamma)} \longrightarrow \mathbb{P}^r$$

by contracting all C_v to $p_{i(v)}$, and the degree and marked points on C are described by d_e and S_v. Hence, we obtain a T-fixed n-pointed stable map of genus g, degree d into $\mathbb{P}^r$ with graph type Γ.

Therefore, this induces a morphism $M_\Gamma \longrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d)^\Gamma$, where $\overline{M}_{g,n}(\mathbb{P}^r, d)^\Gamma$ is the T-fixed stable maps with graph type Γ. So far, we have not accounted the automorphism issue. Let A_Γ acting on M_Γ such that

$$M_\Gamma/A_\Gamma \simeq \overline{M}_{g,n}(\mathbb{P}^r, d)^\Gamma.$$

Notice that A_Γ is given by the following exact sequence of groups:

$$1 \to \prod_{e\in E(\Gamma)} \mathbb{Z}/(d_e) \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1,$$

where $\mathbb{Z}/(d_e)$ is the covering transformation of degree d_e covering on C_e, and $Aut(\Gamma)$ is the automorphism of Γ preserving all labeling.

Before stating the formula for the Euler class of the normal bundle for M_Γ, we need to introduce the notion of flags. A flag is a pair (v, e) such that v is incident to e. Denote $i(F) = i(v)$, and $j(F) = i(v')$, where v' is the other vertex incident to e. The following datum appear in the Euler class of the normal bundle of the fixed loci:

(1)

$$\omega_F := \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e}$$

(2) $\psi_{i(F)}$ is the first chern class of the bundle on M_Γ whose fiber is the contangent space to the component associated to v at $p_{i(F)}$.

For later references, we give the formula for Euler class for M_Γ.

Theorem 2 ([1], Thm.9.2.1). *The reciprocal of Euler class of normal bundle for M_Γ is given by*

$$
\begin{aligned}
\frac{1}{e(N_\Gamma)} = & \prod_{e\in E(\Gamma)} \frac{(-1)^{d_e} {d_e}^{2d_e}}{(d_e)!(\lambda_i-\lambda_j)^{2d_e}} \prod_{a+b=d_e, k\neq i,j} \frac{1}{d_e^{-1}(a\lambda_i+b\lambda_j)-\lambda_k} \\
& \times \prod_{F\in F(\Gamma)} \prod_{j\neq i(F)} (\lambda_{i(F)}-\lambda_j) \prod_{F\in F(\Gamma), n(i(F))\geq 3} \frac{1}{\omega_F-\psi_{i(F)}} \\
& \times \prod_{v\in V(\Gamma)} \prod_{j\neq i(v)} \frac{1}{\lambda_{i(v)}-\lambda_j} \prod_{\mathrm{val}(v)=1, S_v=\emptyset} \omega_{F(v)} \prod_{\mathrm{val}(v)=2, S_v=\emptyset} \frac{1}{\omega_{F_1(v)}+\omega_{F_2(v)}} \\
= & \prod_{e\in E(\Gamma)} \frac{(-1)^{d_e} {d_e}^{2d_e}}{(d_e)!(\lambda_i-\lambda_j)^{2d_e}} \prod_{a+b=d_e, k\neq i,j} \frac{1}{d_e^{-1}(a\lambda_i+b\lambda_j)-\lambda_k} \\
& \times \prod_{F\in F(\Gamma), n(i(F))\geq 3} \frac{1}{\omega_F-\psi_{i(F)}} \\
& \times \prod_{v\in V(\Gamma)} \prod_{j\neq i(v)} (\lambda_{i(v)}-\lambda_j)^{\mathrm{val}(v)-1} \prod_{\mathrm{val}(v)=1, S_v=\emptyset} \omega_{F(v)} \prod_{\mathrm{val}(v)=2, S_v=\emptyset} \frac{1}{\omega_{F_1(v)}+\omega_{F_2(v)}}.
\end{aligned}
$$

3. Localizations on $\mathbb{P}^2$

Consider $\mathbb{P}^2$ with bundle $V := \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Let $\pi : \overline{M}_{0,d+2}(\mathbb{P}^2, d) \longrightarrow \overline{M}_{0,d+1}(\mathbb{P}^2, d)$ be the morphism given by forgetting the last marked point, and $\mathrm{ev}_j : \overline{M}_{0,d+2}(\mathbb{P}^2, d) \longrightarrow \mathbb{P}^2$ be the evaluation at the j -th marked point. Let $U_d = (R^1\pi)_*(\mathrm{ev}_{d+2}^* V)$ be the induced bundle on $\overline{M}_{0,d+1}(\mathbb{P}^2, d)$. On a closed point $f \in \overline{M}_{0,d+1}(\mathbb{P}^2, d)$, the bundle U_d has fiber $\mathrm{H}^1(C, V)$ whose dimension equals

$$
2\dim \mathrm{H}^1(C, \mathcal{O}(-d)) = 2\dim \mathrm{H}^0(C, \mathcal{O}(d-2)) = 2(d-1). \tag{2}
$$

Recall the main theorem (Thm. 1) appeared in the introduction.

$$
I_d := \int_{[\overline{M}_{0,d+1}(\mathbb{P}^2,d)]} c_{2d-2}(U_d) \cup \prod_{i=1}^{d+1} \mathrm{ev}_i^* h^2 = d^{d-2}, \tag{3}
$$

where h is the hyperplane class on $\mathbb{P}^2$. A standard technique to compute I_d is to use Atiyah–Bott localization formula. For this purpose, we introduce a $T = (\mathbb{C}^*)^3$ action on $\mathbb{P}^2$ as in the previous section. A T-fixed point $(C, f) \in \overline{M}_{0,d+1}(\mathbb{P}^2, d)$ gives a labeled graph Γ. Then we only need to compute

$$
I'_d := \sum_{\Gamma\in S_{0,d,d+1}} I'_d(\Gamma) := \sum_{\Gamma\in S_{0,d,d+1}} \frac{1}{|A_\Gamma|} \int_{M_\Gamma} \frac{(c_{2d-2}(U_d) \cup \prod_{i=1}^{d+1} \mathrm{ev}_i^* h^2)|_{M_\Gamma}}{e(N_\Gamma)}. \tag{4}
$$

Taking the non-equivariant limit $\lambda_i \to 0$ of I'_d, which exists, will give I_d.

Following [8], we arrange the weights on V in the following manner. We write $V = \mathcal{O}(-1)_{(1)} \oplus \mathcal{O}(-1)_{(2)}$. Let $\mathcal{O}(-1)_{(k)}|_{p_i}$ has weight $\xi_{i,k}$. Under the standard T-linearization of $\mathcal{O}(1)$, the weight of vector space $\mathcal{O}(1)|_{p_i}$ is λ_i. Since the tautological bundle $\mathcal{O}(-1)$ is the dual bundle of $\mathcal{O}(1)$, the weight of $\mathcal{O}(-1)|_{p_i}$ is $-\lambda_i$. We alter the weight on each piece $\mathcal{O}(-1)_{(k)}$ by regarding $\mathcal{O}(-1)_{(k)}$ as the tensor product of the the tautological bundle and a trivial line bundle $\mathcal{O}$ with equivariant weight λ_k. In this way, the weights of $\mathcal{O}(-1)_{(1)}$ and $\mathcal{O}(-1)_{(2)}$ at p_i are given by the following table.

Points/ Bundle	$\mathcal{O}(-1)_{(1)}$	$\mathcal{O}(-1)_{(2)}$
p_0	$\xi_{0,1} = \lambda_1 - \lambda_0$	$\xi_{0,2} = \lambda_2 - \lambda_0$
p_1	$\xi_{1,1} = 0$	$\xi_{1,2} = \lambda_2 - \lambda_1$
p_2	$\xi_{2,1} = \lambda_1 - \lambda_2$	$\xi_{2,2} = 0$

Now, we compute the equivariant cohomology class of the bundle $c_{2d-2}(U_d)$. Let (C, f) be a T-invariant point in $\overline{M}_{0,d+1}(\mathbb{P}^2, d)$. Consider the normalization sequence for source curve C

$$0 \longrightarrow \mathcal{O}_C \to \bigoplus_{v\in V(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e\in E(\Gamma)} \mathcal{O}_{c_e} \longrightarrow \bigoplus_{F\in F(\Gamma)} \mathcal{O}_{x_F} \longrightarrow 0 \tag{5}$$

Tensorizing with f^*V, and then taking long exact sequence yields:

$$\begin{aligned} c_{2d-2}(U_d)|_{M_\Gamma} &= \prod_{e\in E(\Gamma)} e(\mathrm{H}^1(C_e, V)) \prod_{v\in V(\Gamma)} \mathrm{wt}(V|_{p_{i(v)}})^{\mathrm{val}(v)-1} \\ &= \prod_{k=1}^{2} \prod_{e\in E(\Gamma)} \prod_{t=1}^{d_e-1} \frac{t\xi_{i(v),k} + (d_e - t)\xi_{i(v'),k}}{d_e} \prod_{v\in V(\Gamma)} {\xi_{i(v),k}}^{\mathrm{val}(v)-1}, \end{aligned} \tag{6}$$

where $\xi_{i,k}$ is the weight of $\mathcal{O}(-1)_k$ at p_i.

Lemma 1. *With the above choice of weights, a graph Γ has a non-zero contribution in the localization formula only if the valency of its vertex v with labeling $i(v) = 1$ or 2 is at most 1.*

Proof. Since $\xi_{1,1} = 0, \xi_{2,2} = 0$, if a graph Γ has a vertex with labeling 1 or 2 such that its valency is greater than one, then from (6), the restriction of equivariant class $c_{2d-2}(U_d)$ to the corresponding M_Γ is zero. □

By the lemma, we only have to deal with the following two possibilities.

(a) Γ has only one edge connecting two vertices with labeling 1 and 2.
(b) Γ is a star with central vertex having labeling 0 and all the leaves having labeling 1 or 2.

3.1. **Contribution from Type (a) graphs.** Let $\mathcal{A}(d)$ the the collection of type (a) graph of degree d. By lemma 1, we only need to sum over trees of type (a) and (b). In this subsection, we compute the contribution from type (a) graph.

Proposition 1.

$$\sum_{\Gamma\in\mathcal{A}(d)} I'_d(\Gamma) = \prod_{t=0}^{d} \frac{\lambda_1+\lambda_2}{d^{-1}(t\lambda_1+(d-t)\lambda_2)-\lambda_0} d^{d-2}$$

In particular, if we take the limit $\lambda_0 \to 0$, and set $\lambda_1 = \lambda_2 = \lambda$, then we get:

$$\sum_{\Gamma\in\mathcal{A}(d)} I'_d(\Gamma) = 2^{d+1} d^{d-2}.$$

In the remaining of this section, we will demonstrate Proposition 1. Given a graph Γ of type (a), it has only one edge e of degree d with two vertices v_1, v_2 with $i(v_1) = 1, i(v_2) = 2$ incident to it. Let N_1, N_2 be the numbers of marked points on v, v_2 resepctively. We assume that $N_1, N_2 \geq 3$. Then the reciprocal of the Euler class of the normal bundle is

$$\begin{aligned}\frac{1}{e(N_\Gamma)} = &\frac{(-1)^d d^{2d}}{(d!)^2(\lambda_1-\lambda_2)^{2d}} \times \prod_{t=0}^{d} \frac{1}{d^{-1}(t\lambda_1+(d-t)\lambda_2)-\lambda_0} \\ &\times \prod_{F\in F(\Gamma), n(i(F))\geq 3} \frac{1}{\omega_F - \psi_{i(F)}} \prod_{\mathrm{val}(v)=1, S_v=\emptyset} \omega_{F(v)}.\end{aligned}$$

Here, we only have two flags $F(\Gamma) = \{(v_1, e), (v_2, e)\}$, and the vertices have no contribution since $\mathrm{val}(v_1) = \mathrm{val}(v_2) = 1$.

For the cases $N_i \leq 2$, the formula for the reciprocal of the Euler class are different from above. However, after taking integration over M_Γ, the following formula still valids in these cases.

Proposition 2.

$$\begin{aligned}\int_{M_\Gamma} \frac{1}{e(N_\Gamma)} = &\frac{(-1)^d d^{2d}}{(d!)^2(\lambda_1-\lambda_2)^{2d}} \prod_{t=0}^{d} \frac{1}{d^{-1}(t\lambda_1+(d-t)\lambda_2)-\lambda_0} \\ &\times \left(\frac{d}{\lambda_1-\lambda_2}\right)^{N_1-1} \left(\frac{d}{\lambda_2-\lambda_1}\right)^{N_2-1}\end{aligned}$$

For graphs of type (a), the equivariant cohomology of the bundle U_d is

$$c_{2d-2}(U_d)|_{M_\Gamma} = \prod_{k=1}^{2} \prod_{t=1}^{d-1} \frac{t\xi_{1,k}+(d-t)\xi_{2,k}}{d}.$$

Note that the vertex terms have no contributions since both v_1, v_2 are of valency 1. Since $\xi_{1,1} = 0$, $\xi_{2,1} = \lambda_1 - \lambda_2$, $\xi_{1,2} = \lambda_2 - \lambda_1$, $\xi_{2,2} = 0$, the above formula becomes

$$\begin{aligned} c_{2d-2}(U_d)|_{M_\Gamma} &= \prod_{t=1}^{d-1} \frac{(d-t)(\lambda_1 - \lambda_2)}{d} \prod_{t=1}^{d-1} \frac{t(\lambda_2 - \lambda_1)}{d} \\ &= \frac{(-1)^{d-1}((d-1)!)^2}{d^{2d-2}} (\lambda_1 - \lambda_2)^{2(d-1)} \\ &= \frac{(-1)^{d-1}(d!)^2}{d^{2d}} (\lambda_1 - \lambda_2)^{2(d-1)}. \end{aligned}$$

Combining with the previous calculation on the normal bundle, we have

$$\begin{aligned} \int_{M_\Gamma} \frac{(c_{2d-2}(U_d) \cup \prod_{i=1}^{d+1} \mathrm{ev}_i^* h^2)|_{M_\Gamma}}{e(N_\Gamma)} &= (-1)^{N_2} \frac{\lambda_1^{2(d+1-N_2)} \lambda_2^{2N_2}}{(\lambda_1 - \lambda_2)^{d+1}} d^{d-1} \\ &\quad \times \prod_{t=0}^{d} \frac{1}{d^{-1}(t\lambda_1 + (d-t)\lambda_2) - \lambda_0}. \end{aligned}$$

Now, since two vertices have different labeling, there is no automorphism other than the covering transformations on non-contracted components. Hence $A_\Gamma = \mathbb{Z}/(d\mathbb{Z})$. Note that the summation over all type (a) labeled graphs gives

$$\sum_{N_2=0}^{d+1} (-1)^{N_2} \binom{d+1}{N_2} \lambda_1^{2(d+1)-N_2} \lambda_2^{N_2} = (\lambda_1^2 - \lambda_2^2)^{d+1}.$$

As a result, we obtain

$$\sum_{\Gamma \in \mathcal{A}(d)} I'_d(\Gamma) = (\lambda_1 + \lambda_2)^{d+1} d^{d-2} \prod_{t=0}^{d} \frac{1}{d^{-1}(t\lambda_1 + (d-t)\lambda_2) - \lambda_0}. \tag{7}$$

3.2. **Graphs of Type (b): General Discussion.** In this subsection, we give some general remarks on the computations of the graphs of type (b). We first write down the components occuring in the localization formula. We consider a star-shape with valency k at central vertex v. We order the leaf vertices counterclockwisely by $v_1, \ldots, v_k$. Assume that $i_{v(1)}, \ldots i_{v(s)} = 1$, $i_{v(s+1)}, \ldots, i_{v(k)} = 2$, and the degree of edge e_j incident to v_j is d_j, the numbers of marked points lying on vertex v_j is N_j, on v is N. For simplicity, we denote $i_{v(j)}$ by $i(j)$. They are subject to the condition: $d = \sum_{i=1}^k d_i$, $n = d + 1 = N + \sum_{i=1}^k N_i$.

3.2.1. *Edge Terms.* Since we are dealing with the star-shape, the edge term of a star-shape is just the product of each edge term. For each edge of degree d with vertices labeling 0 and $i \in \{1, 2\}$, the edge contribution from normal bundle is given by

$$\frac{(-1)^d d^{2d}}{(d!)^2 (\lambda_i - \lambda_0)^{2d}} \prod_{t=0}^{d} \frac{1}{d^{-1}(t\lambda_0 + (d-t)\lambda_i) - \lambda_j}, \tag{8}$$

where $j \in \{1,2\} - \{i\}$. While U_d contributes the edge term

$$\prod_{t=1}^{d-1} \frac{t\xi_{0,1} + (d-t)\xi_{i,1}}{d} \prod_{t=1}^{d-1} \frac{t\xi_{0,2} + (d-t)\xi_{i,2}}{d}. \tag{9}$$

- When $i = 1$, $\xi_{1,1} = 0, \xi_{1,2} = \lambda_2 - \lambda_1$, then (9) becomes
$$\frac{(-1)^{d-1}(d-1)!}{d^{d-1}}(\lambda_1 - \lambda_0)^{d-1} \prod_{t=1}^{d-1} \left(\frac{t\lambda_0 + (d-t)\lambda_1}{d} - \lambda_2 \right).$$
Thus, combining with the edge term from normal bundle, we get
$$\frac{d^d}{d!} \frac{1}{(\lambda_1 - \lambda_0)^{d+1}} \frac{1}{\lambda_1 - \lambda_2} \frac{1}{\lambda_2 - \lambda_0}.$$
- When $i = 2$, $\xi_{2,1} = \lambda_1 - \lambda_2$, $\xi_{2,2} = 0$, then (9) gives
$$\frac{(-1)^d d^{2d}}{(d!)^2(\lambda_2 - \lambda_0)^{2d}} \prod_{t=0}^{d} \frac{1}{d^{-1}(t\lambda_0 + (d-t)\lambda_2) - \lambda_1}.$$
Thus, simplifying after multiplying the edge term from normal bundle. we get
$$-\frac{d^d}{d!} \frac{1}{(\lambda_2 - \lambda_0)^{d+1}} \frac{1}{\lambda_1 - \lambda_2} \frac{1}{\lambda_1 - \lambda_0}.$$

Thus, the edge term in the star-shape is given by

$$\begin{aligned}
&\prod_{i=1}^{s} \frac{d_i^{d_i}}{d_i!} \frac{1}{(\lambda_1 - \lambda_0)^{d_i+1}} \frac{1}{(\lambda_1 - \lambda_2)} \frac{1}{(\lambda_2 - \lambda_0)} \\
&\times \prod_{i=s+1}^{k} -\frac{d_i^{d_i}}{d_i!} \frac{1}{(\lambda_2 - \lambda_0)^{d_i+1}} \frac{1}{\lambda_1 - \lambda_2} \frac{1}{\lambda_1 - \lambda_0} \\
&= \frac{(-1)^{k-s}}{(\lambda_1 - \lambda_2)^k} \frac{1}{(\lambda_2 - \lambda_0)^k} \frac{1}{(\lambda_1 - \lambda_0)^k} \prod_{i=1}^{s} \frac{d_i^{d_i}}{d_i!} \frac{1}{(\lambda_1 - \lambda_0)^{d_i}} \prod_{i=s+1}^{k} \frac{d_i^{d_i}}{d_i!} \frac{1}{(\lambda_2 - \lambda_0)^{d_i}}.
\end{aligned}$$

3.2.2. *Vertex Terms.* For such a star-shape with valency k. Since leaf vertex has only valency one, theorem 1 and (6) show that leaf vertex has no contribution. Therefore, for the normal bundle, the vertex term has only contribution at central vertex v:

$$\prod_{j \neq 0} (\lambda_0 - \lambda_j)^{\mathrm{val}(v)-1} = (\lambda_0 - \lambda_1)^{k-1}(\lambda_0 - \lambda_2)^{k-1}.$$

Similarly, vertex terms of U_d has non-trivial contributions only at v:

$$\xi_{0,1}^{\mathrm{val}(v)-1} \xi_{0,2}^{\mathrm{val}(v)-1} = (\lambda_1 - \lambda_0)^{k-1}(\lambda_2 - \lambda_0)^{k-1}.$$

Hence, for such a star-shape, the vertex terms are given by

$$(\lambda_2 - \lambda_0)^{2(k-1)}(\lambda_1 - \lambda_0)^{2(k-1)}.$$

3.2.3. *Flag Terms.* For a star-shape, we have two kinds of flags. One kind of flag is (v_i, e_i) which v_i is the i-th leave vertex and e_i is the edge connecting central vertex v and v_i. We say (v_i, e_i) is the i-th **leaf flag**. Another kind is (v, e_i), and we call this i-th **central flag**.

- Leaf flag with $N_j \geq 2$: the flag $F_j = (v_j, e_j)$ contributes the term
$$\frac{1}{\omega_{F_j} - \psi_{F_j}} = \frac{1}{\frac{\lambda_{i(j)}-\lambda_0}{d_j} - \psi_{i(j)}}$$
appears in the normal bundle. However, by (4), we can evaluate this term by taking integration over M_Γ. In this case, the integration becomes:
$$\int_{\overline{M}_{0,N_j+1}} \frac{1}{\frac{\lambda_{i(j)}-\lambda_0}{d_j} - \psi_{i(j)}} = \sum_{k\geq 0} \int_{\overline{M}_{0,N_j+1}} \left(\frac{d_j}{\lambda_{i(j)} - \lambda_0}\right)^{k+1} \psi_{i(j)}^k.$$
Since $\dim \overline{M}_{0,N_j+1} = N_j - 2$, the integration has value
$$\left(\frac{d_j}{\lambda_{i(j)} - \lambda_0}\right)^{N_j-1}. \tag{10}$$
- Leaf flag with $N_j = 0, 1$: for $N_j = 0$, the flag $F_j = (v_j, e_j)$ contribute the term
$$\omega_{F_j} = \frac{\lambda_{i(j)} - \lambda_0}{d_j}.$$
Notice that for $N_j = 0$, the contribution coincide with the formula (10) under setting $N_j = 0$. Similarly, when $N_j = 1$, there is no term appearing in the formula for the Euler class of normal bundle. Thus, we may regard it as the special case of (10) when $N_j = 1$. Therefore, we can treat all the contribution from the leaf flag for all N_j on equal footing.
- Central flag with $n(v) = k + N \geq 3$: the flag $F'_j = (v, e_j)$ contributes
$$\frac{1}{\omega_{F'_j} - \psi_{F'_j}} = \frac{1}{\frac{\lambda_0-\lambda_i(j)}{d_j} - \psi_{i(j)}}$$
These class are to be integrated on the moduli space of stable curves representing the central vertex $\overline{M}_{0,n(v)}$, which has dimension $n(v) - 3$. Therefore, the contribution from central flag in this case is
$$\begin{aligned}
&\int_{\overline{M}_{0,n(v)}} \prod_{j=1}^{k} \frac{1}{\frac{\lambda_0-\lambda_i(j)}{d_j} - \psi_{i(j)}} \\
=& \int_{\overline{M}_{0,n(v)}} \prod_{j=1}^{k} \left(\sum_{l_j=1}^{\infty} \left(\frac{d_j}{\lambda_0 - \lambda_{i(j)}}\right)^{l_j+1} \psi_{i(j)}^{l_j}\right) \\
=& \sum_{l_1+\cdots+l_k=n(v)-3} \prod_{j=1}^{k} \left(\frac{d_j}{\lambda_0 - \lambda_{i(j)}}\right)^{l_j+1}.
\end{aligned}$$

- Central flag with $n(v) = k + N \leq 2$: if $N \neq 0$, then no additional term appears. On the other hand, if $N = 0$, and $k = 1$, then we get an addition term
$$\omega_{(v,e)} = \frac{\lambda_0 - \lambda_i}{d}.$$
If $N = 0$, and $k = 2$, then we get a new term
$$\frac{1}{\omega_{(v,e_1)} + \omega_{(v,e_2)}} = \frac{1}{d_1^{-1}(\lambda_0 - \lambda_{i(1)}) + d_2^{-1}(\lambda_0 - \lambda_{i(2)})}.$$

3.2.4. *Insertions.* Recall in (4), we have the term
$$\prod_{i=1}^{d+1} (\mathrm{ev}_i^* h^2)_{M_\Gamma}.$$
Then under the standard T-action on $\mathbb{P}^2$, we get
$$\prod_{i=1}^{d+1} (\mathrm{ev}_i^* h^2)_{M_\Gamma} = \lambda_0^{2N} \prod_{j=1}^{s} \lambda_1^{2N_j} \prod_{j=s+1}^{k} \lambda_2^{2N_j}. \tag{11}$$
Hence, multiplying every terms in the corresponding case, we get the explicit formula for each of the summand of (4) for star-shape graph except dividing out the order of automorphism.

3.2.5. *Further Reduction.* First of all, for a given star-shape graph with definite distribution of marked points, we can see that each summand in (4) for a star-shape graph is well-defined when setting $\lambda_0 = 0$. Hence, the summand has non-equivariant limit when we set $\lambda_0 = 0$. Since I'_d has non-equivariant limit I_d and non-equivariant limit $I_d = \lim_{\lambda_i \to 0} I'_d$ is free from $\lambda_i' s$, the non-equivariant limit of I'_d for taking $\lambda_0 = 0$ first then doing the sum will still be I_d. Moreover, if we take $\lambda_0 = 0$, then any type (b) graph with any marked point at central vertex will vanish. Hence, we only need to sum over the type (b) graph with $N = 0$. The formula for each term then becomes

- Edge terms:
$$\frac{(-1)^{k-s}}{(\lambda_1 - \lambda_2)^k} (\lambda_1 \lambda_2)^{-k} \prod_{i=1}^{s} \frac{d_i^{d_i}}{d_i!} \frac{1}{\lambda_1{}^{d_i}} \prod_{i=s+1}^{k} \frac{d_i^{d_i}}{d_i!} \frac{1}{\lambda_2{}^{d_i}}.$$
- Vertex terms:
$$(\lambda_1 \lambda_2)^{2(k-1)}.$$
- Leaf flag terms:
$$\prod_{j=1}^{s} \left(\frac{d_j}{\lambda_1}\right)^{N_j - 1} \prod_{j=s+1}^{k} \left(\frac{d_j}{\lambda_2}\right)^{N_j - 1}.$$
- Central flag terms. Now, since $N = 0$, we divide into three cases:

(1) if $k=1$, then central flag contributes
$$-\frac{\lambda_i}{d},$$
where $i=i(v)$.

(2) if $k=2$, then central flag contributes
$$\frac{-1}{d_1^{-1}\lambda_{i(1)}+d_2^{-1}\lambda_{i(2)}}.$$

(3) if $k\geq 3$, then central flag contributes
$$-\sum_{l_1+\cdots+l_k=k-3}\prod_{j=1}^{k}\left(\frac{d_j}{\lambda_{i(j)}}\right)^{l_j+1}.$$

- Insertions:
$$\prod_{j=1}^{s}\lambda_1^{2N_j}\prod_{j=s+1}^{k}\lambda_2^{2N_j}.$$

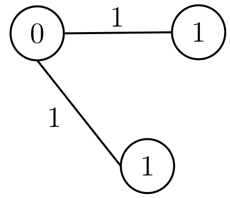
3.3. **Graphs of Type (b): Degree Two.** In this subsection, we verify theorem.1 in the case of degree 2. For a graph of type (b) of degree 2, we have five kinds of graphs.

(1) Γ_1 is an one-edge graph of degree $d=2$ with vertices 0 and 1. The number of marked points is $n=d+1=3$, and they are all lying on the vertex with labeling 1. Then we can compute its contribution by formulae in previous subsections, which is
$$\frac{1}{(\lambda_1-\lambda_2)}\frac{1}{\lambda_1\lambda_2}\frac{2^2}{2!}\frac{1}{\lambda_1^2}\times\frac{2^2}{\lambda_1^2}\times\frac{-\lambda_1}{2}\times\lambda_1^6.$$
Notice that there are no automorphism for the graph preserving vertices in this case. Thus, $A_{\Gamma_1}=\mathbb{Z}/2$. Thus, the final contribution is
$$2\frac{-\lambda_1^2}{(\lambda_1-\lambda_2)\lambda_2}.$$

(2) Γ_2 is an one-edge graph of degree $d=2$ with vertices 0 and 2. There are three marked points, all lying on on the vertex with labeling 2. The contribution is given by
$$\frac{-1}{(\lambda_1-\lambda_2)}\frac{1}{\lambda_1\lambda_2}\frac{2^2}{2!}\frac{1}{\lambda_2^2}\times\frac{2^2}{\lambda_2^2}\times\frac{-\lambda_2}{2}\times\lambda_2^6.$$
Again, there are no automorphism beside edge automorphisms. Hence, $A_{\Gamma_2}=\mathbb{Z}/2$. Therefore, we get
$$2\frac{\lambda_2^2}{(\lambda_1-\lambda_2)\lambda_1}.$$

(3) Γ_3 is a two-edge graph of degree $d=2$ with vertices 0, 1, and 1. It is given by the following figure

Γ_3



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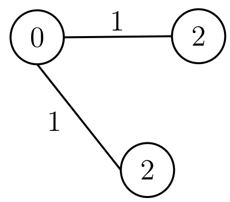
Then the contribution for Γ_3 is given by

$$-\frac{2\lambda_1^2}{(\lambda_1-\lambda_2)^2}.$$

(Note that since $n = d + 1 = 3$, the marked points present at the leaf vertices. Hence, there are no non-trivial automorphism for Γ_3.)

(4) Γ_4 is a two edge graph of degree $d = 2$ with vertices 0, 2, and 2. It is given by the following figure

Γ_4



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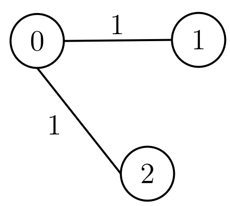
Again, there are no non-trivial automorphism for Γ_4. The contribution for Γ_4 is given by

$$\frac{-2\lambda_2^2}{(\lambda_1-\lambda_2)^2}.$$

(5) Γ_5 is a two-edge graph of degree 2 with vertices $0, 2, 2$. It is given by the following figure

Γ_5



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There are no non-trivial automorphism for Γ_5 as well, and summation over all distribution of marked points is given by

$$\frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}.$$

If we divide all five possibilities into two groups by the valencies, then the contribution from valency 1 graph is given by

$$\frac{-2\lambda_1^2}{(\lambda_1-\lambda_2)\lambda_2}+\frac{2\lambda_2^2}{)\lambda_1-\lambda_2)\lambda_1}=\frac{2(\lambda_2^3-\lambda_1^3)}{(\lambda_1-\lambda_2)\lambda_1\lambda_2}=-2\frac{\lambda_2^2+\lambda_1\lambda_2+\lambda_1^2}{\lambda_1\lambda_2}.$$

We put $\lambda_1 = \lambda_2 = \lambda$, the contribution of valency 1 graph is -6.

On the other hand, for valency 2 graph, we have

$$\frac{-2\lambda_1^2}{(\lambda_1-\lambda_2)^2}+\frac{-2\lambda_2^2}{(\lambda_1-\lambda_2)^2}+\frac{(\lambda_1+\lambda_2)^2}{\lambda_1-\lambda_2)^2}=\frac{-(\lambda_1-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}=-1.$$

By (7), and put $\lambda_1=\lambda_2=\lambda$, $\lambda_0=0$, we get

$$(2\lambda)^3\frac{1}{\lambda^3}=8.$$

Hence, $I_2'=-6-1+8=1$. This coincide with $2^{2-2}=2^0=1$. Therefore, we verified (3) for the case $d=2$.

Moreover, it is interesting to note that if we sum over terms for the labeld graph with the same valencies, then we can cancel the power of $(\lambda_1-\lambda_2)$ appearing in the denominator. Therefore, we can set $\lambda_1=\lambda_2=\lambda$ and get the a number. Adding these numbers gives I_d. To see this, we think each summand of (4) as a rational function $f_i(\lambda_1,\lambda_2)/g_i(\lambda_1,\lambda_2)$, and

$$I_d'(\lambda_1,\lambda_2)=\sum_{i=1}^{N}\frac{f_i(\lambda_1,\lambda_2)}{g_i(\lambda_1,\lambda_2)}.$$

Then $I_d=\lim_{\lambda_i\to 0}\sum_{i=1}^{N}\frac{f_i(\lambda_1,\lambda_2)}{g_i(\lambda_1,\lambda_2)}$. Now, we let the rational function $F_j(\lambda_1,\lambda_2)/G_j(\lambda_1,\lambda_2)$ be the sum of f_i/g_i over all graph Γ in $S_{0,n,d}$ of fixed valency. If we know that after canceling the common factor of F_j and G_j such that $G_j(\lambda,\lambda)\neq 0$, then the expression $F_j(\lambda,\lambda)/G_j(\lambda,\lambda)$ is well-defined. Moreover, in the case of degree 2, we have $F_i(\lambda,\lambda)/G_i(\lambda,\lambda)=a_i(d)$, where $a_i(d)$ involves only d and it is free of λ. Thus,

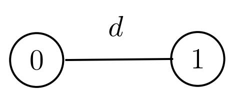
$$I_d=\lim_{\lambda\to 0}I_d'(\lambda,\lambda)=\lim_{\lambda\to 0}\sum_{j=1}^{m}\frac{F_j(\lambda,\lambda)}{G_j(\lambda,\lambda)}=\lim_{\lambda\to 0}\sum_{j=1}^{m}a_j(d)=\sum_{j=1}^{m}a_j(d).$$

In the next subsection, we show that summation over valency 2 graph with degree d gives a rational function $F(\lambda_1,\lambda_2)/G(\lambda_1,\lambda_2)$ with $G(\lambda,\lambda)\neq 0$.

3.4. **Graphs of Type (b): Valency One and Two.** In this section, we compute the summation of localization data over degree d type (b) graphs of valency one and two.

3.4.1. *Valency One.* For valency one type (b) graphs of degree d, we have the following two possibilities.

(1) Γ_1 is an one-edge graph with vertices $0,1$. All the marked points $\{1,\dots,d+1\}$ lie on the vertex with labeling 1.

Γ_1 

From the formulae in section 2.2, we have

- Edge terms:

$$\frac{1}{(\lambda_1-\lambda_2)}\frac{1}{\lambda_1\lambda_2}\frac{d^d}{d!}\frac{1}{\lambda_1^d}.$$

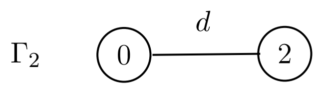
- Vertex terms: 1.
- Leaf flag terms: $\frac{d^d}{\lambda_1^d}$.

- Central flags: $\frac{-\lambda_1}{d}$.
- Insertions: $\lambda_1^{2(d+1)}$.
- Automorphisms: As in the case of $d = 2$, $A_\Gamma = \mathbb{Z}/d$. Hence, $|A_{\Gamma_1}| = d$.

Combining together, we get

$$\frac{-\lambda_1^2}{(\lambda_1 - \lambda_2)\lambda_2} \frac{d^{2(d-1)}}{d!}.$$

(2) Γ_2 is an one edge-graph with vertices $0, 2$. All the marked points $\{1, \ldots, d+1\}$ lie on the vertex with labeling 2.



Similar to Γ_1, we can obtain

- Edge terms:

$$\frac{-1}{(\lambda_1 - \lambda_2)} \frac{1}{\lambda_1\lambda_2} \frac{d^d}{d!} \frac{1}{\lambda_2^d}.$$

- Vertex terms: 1.
- Central flags: $\frac{-\lambda_2}{d}$.
- Insertions: $\lambda_2^{2(d+1)}$.
- Automorphisms: By the same token as in Γ_1, we have $|A_{\Gamma_2}| = d$.

Multiplying together, we get

$$\frac{\lambda_2^2}{(\lambda_1 - \lambda_2)\lambda_1} \frac{d^{2(d-1)}}{d!}.$$

As a result, the contribution of summation over valency 1 graph is

$$\frac{\lambda_2^3 - \lambda_1^3}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)} \frac{d^{2d-2}}{d!} = -\frac{\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2}{\lambda_1\lambda_2} \frac{d^{2d-2}}{d!}. \tag{12}$$

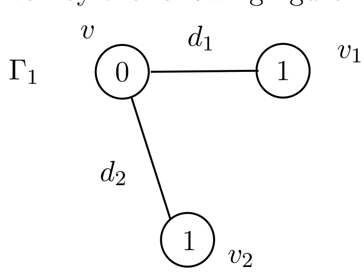
If we set $\lambda_1 = \lambda_2 = \lambda$, (12) becomes

$$-\frac{3}{d!} d^{2d-2}. \tag{13}$$

In the case $d = 2$, we get -6, which coincide with our previous calculation.

3.4.2. *Valency Two.* For valency two type (b) graphs of degree d, there are four possibilities.

(1) Γ_1 is a two edge graph given by the following figure



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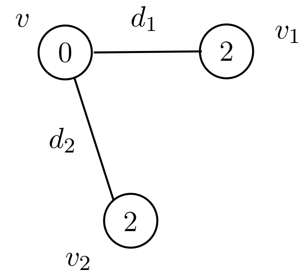
where $d_1 + d_2 = d$ and $d_1 \geq d_2$. We let N_i be the number of marked points lying on v_i, and $N_1 + N_2 = d + 1$.

- Edge terms:
$$\frac{1}{(\lambda_1-\lambda_2)^2}\frac{1}{(\lambda_1\lambda_2)^2}\frac{d_1^{d_1}}{d_1!}\frac{d_2^{d_2}}{d_2!}\frac{1}{\lambda_1^d}.$$
- Vertex terms: $(\lambda_1\lambda_2)^2$.
- Central flag term:
$$\frac{1}{\lambda_1/d_1+\lambda_1/d_2}=\frac{-d_1d_2}{d\lambda_1}.$$
- Insertions and leaf flag terms: For fixed (N_1,N_2), the insertions give
$$\lambda_1^{2N_1}\lambda_1^{2N_2}=\lambda_1^{2(d+1)},$$
while the leaf flag terms contribute
$$\left(\frac{d_1}{\lambda_1}\right)^{N_1-1}\left(\frac{d_2}{\lambda_1}\right)^{N_2-1}.$$
Then we have
$$\left(\frac{d_1}{\lambda_1}\right)^{N_1-1}\left(\frac{d_2}{\lambda_1}\right)^{N_2-1}\times\lambda_1^{2N_1}\lambda_1^{2N_2}=\frac{\lambda_1^2}{(d_1d_2)^2}(d_1\lambda_1)^{N_1}(d_2\lambda_1)^{N_2}.$$
Also, there are $\binom{d+1}{N_1}$ ways to distribute marked points on N_1-many marked points on v_1 and N_2 many on v_2. Therefore, summation over N_1 from 0 to $d+1$, we get
$$\begin{aligned}&\frac{\lambda_1^2}{d_1d_2}\sum_{N_1=0}^{d+1}\binom{d+1}{N_1}(d_1\lambda_1)^{N_1}(d_2\lambda_1)^{N_2}\\=&\frac{\lambda_1^2}{d_1d_2}(d_1\lambda_1+d_2\lambda_1)^{d+1}\\=&\frac{\lambda_1^2}{d_1d_2}d^{d+1}\lambda_1^{d+1}.\end{aligned}$$
However, when $d_1=d_2=d/2$, the same graph will be appear twice in above summation, we need to divide it by 2.
- Autmorphism: Since $N_1+N_2=d+1$, and hence there are at least one marked point appearing at one leaf vertices. Thus, $|A_{\Gamma_1}|=d_1d_2$.

In summary, total contribution from the graph of type Γ_1 is
$$\frac{-\lambda_1^2}{(\lambda_1-\lambda_2)^2}\frac{d_1^{d_1-1}d_2^{d_2-1}}{d_1!d_2!}d^d. \tag{14}$$
When $d_1=d_2$, we need to divide above by 2.

(2) Γ_2 is a two-edge graph given by



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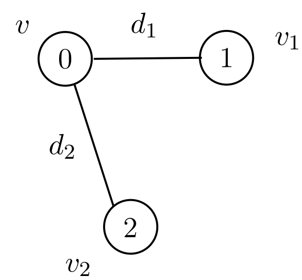
where again $d_1 + d_2 = d$, and $d_1 \geq d_2$. Let N_i be the number of marked points lying on v_i, and $N_1 + N_2 = d + 1$. The formula of each term for Γ_2 is symmetric to Γ_1 by exchanging λ_1 and λ_2. Hence, the total contribution for summation over graph Γ_2 is

$$\frac{-\lambda_2^2}{(\lambda_1 - \lambda_2)^2} \frac{d_1^{d_1-1}}{d_1!} \frac{d_2^{d_2-1}}{d_2!} d^d. \tag{15}$$

By the same token as Γ_1, when $d_1 = d_2$, we need to correct the contribution by dividing 2 in (15).

(3) Γ_3 is given by



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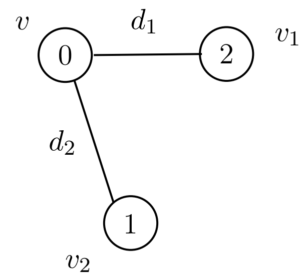
where $d_1 + d_2 = d$, $d_1 \geq d_2$.

- Edge terms:
$$\frac{-1}{(\lambda_1 - \lambda_2)^2} \frac{1}{(\lambda_1 \lambda_2)^2} \frac{d_1^{d_1}}{d_1!} \frac{1}{\lambda_1^{d_1}} \frac{d_2^{d_2}}{d_2!} \frac{1}{\lambda_2^{d_2}}.$$
- Vertex terms: $(\lambda_1 \lambda_2)^2$
- Central flag terms:
$$\frac{-1}{\lambda_1/d_1 + \lambda_2/d_2} = \frac{-d_1 d_2}{d_2 \lambda_1 + d_1 \lambda_2}.$$
- Insertions and leaf flag terms: By the same argument in previous case, the contribution of summation over all distribution of marked points, we have
$$\frac{\lambda_1 \lambda_2}{(d_1 d_2)^2} (d_1 \lambda_1 + d_2 \lambda_2)^{d+1}.$$

Hence, the contribution for graph of shape Γ_3 is

$$\frac{(d_1 \lambda_1 + d_2 \lambda_2)^{d+1}}{(\lambda_1 - \lambda_2)^2 \lambda_1^{d_1-1} \lambda_2^{d_2-1} (d_2 \lambda_1 + d_1 \lambda_2)} \times \frac{d_1^{d_1-1} d_2^{d_2-1}}{d_1! d_2!}. \tag{16}$$

(4) Γ_4 is given by



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where $d_1 + d_2 = d, d_1 \geq d_2$. The contribution of Γ_4 is given by exchanging λ_1 and λ_2 in (16). Hence, we have

$$\frac{(d_2\lambda_1 + d_1\lambda_2)^{d+1}}{(\lambda_1 - \lambda_2)^2 \lambda_1^{d_2-1} \lambda_2^{d_1-1} (d_1\lambda_1 + d_2\lambda_2)} \times \frac{d_1^{d_1-1} d_2^{d_2-1}}{d_1! d_2!}. \tag{17}$$

We add up (14),(15),(16),(17), and we get

$$\begin{aligned} &\frac{d_1^{d_1-1} d_2^{d_2-1}}{d_1! d_2!} \frac{1}{(\lambda_1 - \lambda_2)^2 (\lambda_1\lambda_2)^{d-1} (d_1\lambda_1 + d_2\lambda_2)(d_2\lambda_1 + d_1\lambda_2)} \\ &\times \{(d_1\lambda_1 + d_2\lambda_2)^{d+2} \lambda_1^{d_2} \lambda_2^{d_1} + (d_2\lambda_1 + d_1\lambda_2)^{d+2} \lambda_1^{d_1} \lambda_2^{d_2} \\ &\quad - d^d (\lambda_1\lambda_2)^{d-1} (d_1\lambda_1 + d_2\lambda_2)(d_2\lambda_1 + d_1\lambda_2)(\lambda_1^2 + \lambda_2^2)\} \end{aligned} \tag{18}$$

Put $z = \lambda_1/\lambda_2$ in (18), then (18) becomes

$$\begin{aligned} &\frac{d_1^{d_1-1} d_2^{d_2-1}}{d_1! d_2!} \frac{1}{(z-1)^2 z^{d-1} (d_1 z + d_2)(d_2 z + d_1)} \\ &\times (z^{d_2} (d_1 z + d_2)^{d+2} + z^{d_1} (d_2 z + d_1)^{d+2} - d^d (z^2+1) z^{d-1} (d_1 z + d_2)(d_2 z + d_1)). \end{aligned} \tag{19}$$

Let $G(z) := z^{d_2} (d_1 z + d_2)^{d+2} + z^{d_1} (d_2 z + d_1)^{d+2} - d^d (z^2+1) z^{d-1} (d_1 z + d_2)(d_2 z + d_1))$. Then one computes

$$G(1) = G'(1) = 0;\ G''(1) = 2d_1 d_2 (d-4) d^d.$$

Hence, the numerator in (19) can cancel the term $(z-1)^2$ in the denominator in (19). As a result, taking $z = 1$ in (19) after caneling $(z-1)^2$, we get

$$\frac{d_1^{d_1-1} d_2^{d_2-1}}{d_1! d_2!} \frac{2d_1 d_2 (d-4) d^d}{2! d^2} = \frac{d_1^{d_1} d_2^{d_2}}{d_1! d_2!} (d-4) d^{d-2}$$

(Notice that $G(z) = \frac{(z-1)^2}{2!} G''(1) + O((z-1)^3)$).

Hence, for $d_1 \neq d_2$, the total contribution from valency two graph is given by

$$\frac{d_1^{d_1} d_2^{d_2}}{d_1! d_2!} (d-4) d^{d-2}. \tag{20}$$

When $d_1 = d_2$, we do not need to distinguish the case Γ_3 and Γ_4. Each graph with leaf vertices 1 and 2 is counted twice in the above summation. Hence we need to correct them by dividing 2. Also, we know that we need to divide 2 in the case of Γ_1 and Γ_2. Therefore, when $d_1 = d_2$, the correct contribution for valency two graph is given by

$$\frac{d_1^{d_1} d_2^{d_2}}{d_1! d_2!} \frac{(d-4)}{2} d^{d-2}. \tag{21}$$

When we put $d_1 = d_2 = 1$, $d = d_1 + d_2 = 2$, (21) gives -1, which coincides with our previous calculation for $d = 2$.

3.5. **Graphs of Type (b): Degree Three.** In this subsection, we calculate I'_d for the case $d = 3$. As before, we calculate contributions from each valency and add them up. For valency one graphs of type (b), (13) gives

$$-\frac{3}{3!} 3^{6-2} = -\frac{81}{2}.$$

While for valency two graphs, only ordered partition (d_1, d_2) of $d = 3$ is $(2, 1)$. Hence, (20) gives

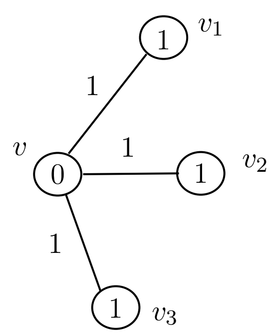
$$\frac{2^2}{2!}(3-4)3^{3-2} = -6.$$

For type (a) graph, setting $d = 3$ and $\lambda_1 = \lambda_2 = \lambda$, and $\lambda_0 = 0$ in (7) gives

$$3(2\lambda)^4/\lambda^4 = 48.$$

It remains to calculate the contribution from valency 3 graphs of type (b). There are four possibilities

(1) Leaf vertices are $(1, 1, 1)$:

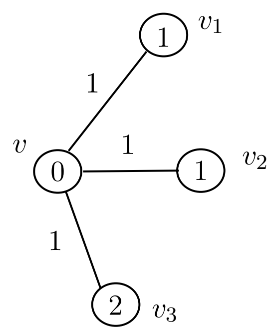


- Edge terms:
$$\frac{1}{(\lambda_1 - \lambda_2)^3} (\lambda_1 \lambda_2)^{-3} \frac{1}{\lambda_1^3}.$$
- Vertex terms: $(\lambda_1 \lambda_2)^4$.
- Central flag terms: $-\frac{1}{\lambda_1^3}$
- Insertions and leaf flag terms: For fixed (N_1, N_2, N_3), where N_i is the number of marked points on the vertex v_i. The insertions gives
$$\lambda_1^{2N_1} \lambda_1^{2N_2} \lambda_1^{2N_3}.$$
While the leaf vertices contributes
$$\left(\frac{1}{\lambda_1}\right)^{N_1 - 1} \left(\frac{1}{\lambda_1}\right)^{N_2 - 1} \left(\frac{1}{\lambda_1}\right)^{N_3 - 1}.$$

Together, we have
$$\lambda_1^3\lambda_1^{N_1}\lambda_1^{N_2}\lambda_1^{N_3}.$$
The numbers of ways to divide $d+1$ elements into three groups is just
$$\binom{d+1}{N_1}\binom{d+1-N_1}{N_2}\binom{d+1-N_1-N_2}{N_3} = \frac{(d+1)!}{N_1!N_2!N_3!}.$$
We summation over all possible (N_1, N_2, N_3), and we get
$$\lambda_1^3 \sum_{N_1+N_2+N_3=4} \frac{(d+1)!}{N_1!N_2!N_3!}\lambda_1^{N_1}\lambda_1^{N_2}\lambda_1^{N_3} = (3\lambda_1)^{d+1} = 81\lambda_1^7.$$
Therefore, after summation over all distribution of marked points, we have:
$$\frac{-81\lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)^3}.$$
Again, here we label the vertices by v_1, v_2, v_3, regarding them as distinct vertices. However, we actually counted the same graph (forgetting the labeling v_1, v_2, v_3) 3! times. Therefore, to get the correct contribution in the localization formula, we need to divide by 3!. Thus, the correct contribution for this case is

(22)
$$\frac{-27\lambda_1^2\lambda_2}{2(\lambda_1-\lambda_2)^3}$$

(2) The leaf vertices are $(1, 1, 2)$:



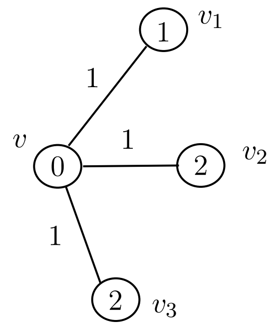
- Edge terms:
$$-\frac{1}{(\lambda_1-\lambda_2)^3}(\lambda_1\lambda_2)^{-3}\frac{1}{\lambda_1^2}\frac{1}{\lambda_2}$$
- Vertex terms: $(\lambda_1\lambda_2)^4$
- Central flag terms: $-\frac{1}{\lambda_1^2\lambda_2}$
- Insertions and leaf flag terms: By the same argument in previous case, after summation over all possible (N_1, N_2, N_3), we get
$$\lambda_1^2\lambda_2 \sum_{N_1+N_2+N_3=4} \frac{(d+1)!}{N_1!N_2!N_3!}\lambda_1^{N_1}\lambda_1^{N_2}\lambda_2^{N_3} = \lambda_1^2\lambda_2(2\lambda_1+\lambda_2)^4.$$

The summation over all N_i gives
$$\frac{(2\lambda_1+\lambda_2)^4}{(\lambda_1-\lambda_2)^3\lambda_1}.$$

Similar to previous case, we need to divide by 2!1! in the final outcome.

$$\frac{(2\lambda_1+\lambda_2)^4}{2(\lambda_1-\lambda_2)^3\lambda_1}. \tag{23}$$

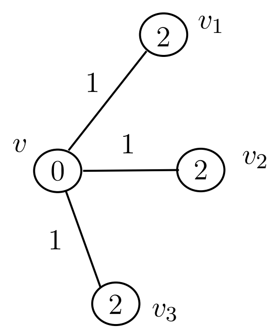
(3) The leaf vertices are $(1,2,2)$:



The formula for case (3) is symmetric to (2) by exchanging the role of λ_1 and λ_2. Therefore, it is given by

$$-\frac{(\lambda_1+2\lambda_2)^4}{2(\lambda_1-\lambda_2)^3\lambda_2}. \tag{24}$$

(4) The leaf vertices are $(2,2,2)$:



The contribution for case (4) is symmetric to (1) by exchanging the role of λ_1 and λ_2. Hence it is given by

$$\frac{27\lambda_1\lambda_2^2}{2(\lambda_1-\lambda_2)^3}. \tag{25}$$

We add up (22), (23),(24), (25) together, and we get

$$\frac{-27\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)+\lambda_2(2\lambda_1+\lambda_2)^4-\lambda_1(\lambda_1+2\lambda_2)^4}{2\lambda_1\lambda_2(\lambda_1-\lambda_2)^3}. \tag{26}$$

Put $\lambda_1=z, \lambda_2=1$, and let $G(z)=-27z^2(z-1)+(2z+1)^4-z(z+2)^4$, then (26) becomes

$$\frac{G(z)}{2z(z-1)^3}. \tag{27}$$

One can verify that

$$G(z)=-(z-1)^3(z^2-5z+1),$$

and this cancel with $(z-1)^3$ in the denominator. Thus, the contribution from valency 3 graph is given by setting $z = 1$ in (27), which is $\frac{3}{2}$. As a result, the total contribution of summation over all valencies is

$$\frac{-81}{2} - 6 + 48 + 3/2 = -45 + 48 = 3.$$

This verifies the theorem for $d = 3$.

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